

Symmetric Equilibria in Stochastic Timing Games

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Abstract

We construct subgame-perfect equilibria with mixed strategies for symmetric stochastic timing games with arbitrary strategic incentives. The strategies are qualitatively different for local first- or second-mover advantages, which we analyse in turn. When there is a local second-mover advantage, the players may conduct a war of attrition with stopping rates that we characterize in terms of the Snell envelope from the general theory of optimal stopping, which is very general but provides a clear interpretation. With a local first-mover advantage, stopping typically results from preemption and is abrupt. Equilibria may differ in the degree of preemption, precisely at which points it is triggered. We provide an algorithm to characterize where preemption is inevitable and to establish the existence of corresponding payoff-maximal symmetric equilibria.

Keywords: Stochastic timing games, mixed strategies, subgame perfect equilibrium, optimal stopping, Snell envelope.

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1 Introduction

The aim of this paper is to construct and understand subgame-perfect equilibria in symmetric stochastic timing games, which have important applications, for instance, in strategic real option models. It is well known that in many timing games in continuous time there exist no equilibria in pure strategies. If they do exist, however, they typically involve asymmetric payoffs that only depend on the respective roles of the players, which must be determined before the game starts. Then there is an unresolved strategic conflict. Here we strive for a rather general existence result and possibly symmetric payoffs, so we consider mixed strategies.

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In particular, no assumption is made concerning the local incentives, which can move randomly between first- and second-mover advantages.

Restricting attention to games with a second-mover advantage is known to be helpful for equilibrium existence. We begin by analyzing that case, too, demonstrating the general payoff asymmetry in pure-strategy equilibria. Our main contribution for this case is the construction of mixed strategy equilibria with symmetric payoffs in a possibly general model, making no specific assumptions concerning the underlying uncertainty. Nevertheless, the equilibrium strategies have a clear characterization and interpretation, using the concept of the Snell envelope from optimal stopping theory. Specifically, we can describe the stopping rates that make a player indifferent to stay in the game when forgoing a profitable local payoff. Due to the possible uncertainty, the tradeoffs can be all in expected terms. Since we are not assuming any kind of smoothness or monotonicity of the underlying payoff processes, we also generalize existing results for purely deterministic models.

With a first-mover advantage, there is often a preemption incentive that leads to equilibrium existence problems even with mixed strategies in very simple, completely well-behaved deterministic models.¹ One needs to extend strategies to model preemption appropriately in continuous time, which requires some coordination mechanism (not to be confused with public *correlation*). As there is no respective “next period”, the players have to be able to ensure that the game ends instantaneously regardless, but without simultaneously stopping to occur with probability 1. We use the generalization of the concept of Fudenberg and Tirole (1985) to stochastic models provided in Riedel and Steg (2014), which gives us preemption equilibria that correspond to symmetric mixed equilibria in discrete time. These can be combined with the previous equilibria for local second-mover advantages with continuous strategies, to obtain a general existence and characterization of subgame-perfect equilibria with symmetric payoffs without any restriction on the order of payoff processes.

These equilibria now allow for richer models of strategic real options, for instance, which typically focus on preemption. In the latter case one needs to ensure that waiting until preemption starts is optimal, however, which may hold in a specific model or requires a particular assumption.² Here we may have some continuous stopping beforehand.

Depending on the profitability of future continuation equilibria, preemption does not have to be triggered just because there is a local first-mover advantage. As the term says, preemption destroys future continuation payoffs; so the less often preemption occurs, the higher the resulting equilibrium payoffs. To determine at which times preemption is indeed inevitable, we provide an algorithm working under the additional assumption that simultaneous stopping is generally the worst outcome (weakly). Then we find that in any equilibrium with symmetric

¹See, e.g., Fudenberg and Tirole (1985) and Hendricks and Wilson (1992).

²See Riedel and Steg (2014) on such properties. For a strategic investment model where both preemption and attrition situations can arise see Steg and Thijssen (2015).

payoffs in every subgame, the equilibrium payoff can never exceed the expected value of optimally stopping the *minimum* of the local leader and follower payoffs. That means, no matter how the players mix, possibly even with arbitrary public correlation and independently of infinite remaining time, they can never benefit from a high value of the underlying payoff processes if that is not attained by both the leader and follower payoffs simultaneously.

Then we know that the game has to end by preemption whenever the leader payoff exceeds the equilibrium payoff bound. Iterating this procedure cumulates in the identification of times when preemption cannot be avoided. Confining preemption to those times, we obtain an equilibrium with least sustainable preemption and highest possible payoffs.

1.1 Main theorems

We use the formal concept of subgame-perfect equilibria with mixed strategies for timing games developed in Riedel and Steg (2014), who argue that subgames are appropriately identified by stopping times (the latter are feasible decision nodes, but cannot be represented by considering deterministic times only). Mixed strategies take the form of distribution functions over time, that can react to the dynamic exogenous information about the state of the world. We further apply the mentioned strategy extensions for preemption regimes. Uncertainty may affect the underlying payoffs, but may also (just) represent public correlation devices. Given that framework, this paper develops three main Theorems 5.3, 7.3 and 8.2, which build on each other.

Theorem 5.3 constructs subgame-perfect equilibria with mixed strategies for games with a systematic (weak) second-mover advantage. Therefore the players have to coordinate on an appropriate payoff process, which consists of the leader payoff up to some feasible point, where there is either a simultaneous stopping equilibrium, or which is sufficiently late such that both players will then have stopped for sure. The equilibrium proceeds by optimally stopping this fixed process. As long as there are expected gains, no player stops. If a point is reached, however, where it would be strictly optimal to stop – i.e., any delay would imply a loss – then there has to be a compensation in terms of some probability to obtain the higher follower payoff. We characterize the exact rate that the respective opponent has to use to make each player indifferent at such points.

Owing to its generality, this result is technically not as clear as its interpretation. With typical Brownian models, for instance, one cannot apply local arguments as there is no path monotonicity at all. Consequently, it is then also not possible to distinguish proper time intervals on which mixing occurs – imagine a Brownian motion fluctuating around the boundary of the region where mixing indeed takes place. Nevertheless, using martingale arguments we obtain a clear representation of strategies involving the concept of Snell envelope from the theory of optimal stopping, which allows us to speak meaningfully of a (local) expected

loss, for instance. These strategies will typically be continuous up to some terminal jumps. Another important question is then time consistency. If we define mixed strategies for all subgames, i.e., stopping times, we have to ensure that they imply consistent conditional stopping probabilities throughout the game, which is generally not trivial.

Theorem 7.3 then makes use of the mentioned strategy extensions, which allow us to provide symmetric preemption equilibria for regimes with a first-mover advantage. The theorem establishes that they form feasible continuation equilibria when leaving regimes with second-mover advantages. In aggregate we thus obtain payoff-symmetric equilibria for games without any restriction on the local incentives. There may be arbitrary, random alternations of first- or second-mover advantages.

Theorem 8.2 determines *efficient* symmetric equilibria. While the previous ones involve extreme preemption – whenever there is a strict first-mover advantage – we now identify equilibria with least sustainable preemption, resulting in the highest feasible payoffs. For that purpose we focus on *payoff-symmetric* equilibria, with symmetric payoffs in every subgame, since this property has important implications for equilibrium strategies. Roughly, conditional stopping probabilities can only differ when players are currently indifferent to become leader or follower. With the additional assumption that simultaneous stopping is not strictly better than leading or following, in equilibrium the players can coordinate at most on optimally stopping the minimum of the leader and follower payoff processes. Whenever the leader payoff exceed that value, there must be preemption. Knowing this restricts the relevant stopping times in the previous problem, which further reduces the attainable value. Iterating the procedure formally as an algorithm identifies inevitable preemption points.

Theorem 8.2 establishes that we do obtain a well-defined equilibrium in the end, not only a limit value. It is based on the previous equilibria, but suppressing preemption where possible. The main problem is to show that there remain proper equilibria where preemption does take place, and that we indeed have well measurable, time-consistent strategies when applying the proposed algorithm to all subgames.

1.2 Related literature

Strategic timing problems appear in an abundance of contexts, in particular in economics but also in biology, e.g., and there is a vast related literature.

On the one hand there is a branch on deterministic timing problems in continuous time addressing a wide range of applications, where typically a distinction is made between preemption models and wars of attrition. Correspondingly, Hendricks and Wilson (1992) and Hendricks et al. (1988) study stylized models with systematic first- and second-mover advantages, respectively. A war of attrition appears in Ghemawat and Nalebuff (1985) who consider

exit from a declining industry.³ In a seminal contribution, Fudenberg and Tirole (1985) emphasize subgame-perfection in a symmetric preemption game. Hoppe and Lehmann-Grube (2005) model a similar technology adoption game, allowing the leader payoff function to be multi-peaked while restricting the follower payoff to be nonincreasing.⁴ Without uncertainty, these games proceed quite linearly due to perfect foresight. More complications arise when the incentives may vary more freely. Laraki et al. (2005) consider general deterministic N -player games with payoffs that are just continuous functions of time (for given identities of first-movers). They prove that there do always exist ε -equilibria, but not necessarily exact equilibria.

On the other hand there is also a wide branch of the literature considering (continuous-time) timing games with uncertainty. Dutta and Rustichini (1993), e.g., formulate a symmetric Markovian setting. However, restricting themselves to pure strategies, their Markov perfect equilibrium payoffs are generally asymmetric.

Important applications with uncertainty are strategic real options. An early contribution is Smets (1991). A typical symmetric model of preemptive investment is that of Mason and Weeds (2010).⁵ Weeds (2002) considers strategic irreversible R&D investment, while Murto (2004) studies exit from a duopoly.

Finally, as we emphasize uncertainty, the literature on Dynkin games with its large tradition has to be named. As these are two-person, *zero-sum* timing games, the classical question is the existence of an equilibrium saddle point, or value, under varying conditions. We here just refer to the more recent work by Touzi and Vieille (2002), since their payoff processes are very general and – more importantly – since they introduce another concept of mixed strategies (but without consideration of subgames). Touzi and Vieille (2002) prove that many more Dynkin games have a value if one allows for such mixed strategies.

Recently, also some more abstract work considering stochastic timing games with non-zero-sum payoffs has been conducted. Hamadène and Zhang (2010), e.g., prove existence of Nash equilibrium for 2-player games with a general second-mover advantage.⁶

³Fudenberg and Tirole (1986) analyse a market exit problem with incomplete information. Bulow and Klemperer (1999) consider a similar problem with more than two firms.

⁴Some implication of F being a supermartingale will be discussed here in Section 4. Dutta et al. (1995) obtain again a similar structure as Fudenberg and Tirole (1985) (including the single-peakedness assumption) from a model of product differentiation.

⁵Pawlina and Kort (2006) consider a similar model with asymmetric investment costs and Thijssen (2010) one with firm-specific uncertainty. Lambrecht and Perraudin (2003) model preemption with incomplete information.

⁶See also Hamadène and Hassani (2014) for an extension to N players using a similar approach. Laraki and Solan (2013) make less assumptions concerning the incentives in a 2-player game. Consequently, even allowing for mixed strategies, they can only prove existence of ε -equilibria.

1.3 Outline

This paper is organized as follows. In Section 2 we define our timing games, making only minimal regularity assumptions, and we introduce the concept of subgame-perfect equilibria in mixed strategies as developed in Riedel and Steg (2014).

Although we are generally working with mixed strategies, equilibrium verification is related to solving optimal stopping problems by linearity, for which we establish a convenient representation in Section 3. There we also present the needed facts from the general theory of optimal stopping. On the one hand, strategies will be represented by the Snell envelope, which we motivate. On the other hand, in our games we have to be quite careful about existence of optimal stopping times, which depends strongly on path properties of the involved processes, so we will address some details.

By a first application of this theory in Section 4 we establish equilibria in pure strategies and argue that they typically generate coordination problems. These are resolved in Section 5 by the construction of subgame-perfect equilibria in mixed strategies, in a first step for games with a systematic second-mover advantage. Although our representation of the equilibrium strategies can be well interpreted, we derive a completely explicit equilibrium for a market exit example in Section 6.

In Section 7 we use the mentioned strategy extensions to obtain equilibria in regimes with first-mover advantages, which then enables us to construct and characterize subgame-perfect equilibria for arbitrary symmetric stopping games. Finally we identify equilibria with maximal payoffs and least possible preemption in Section 8. Section 9 concludes. The appendix contains some technical results and the proofs.

2 The timing game

We use the framework for subgame-perfect equilibria with mixed strategies developed in Riedel and Steg (2014), where the concepts summarized in this section are explained in more detail. Here we only consider symmetric games, which allows some simplifications incorporated in the following.

The timing game consists of two players $i = 1, 2$, who each decide when to stop in continuous time $t \in [0, \infty]$. However, there is uncertainty about the state of the world, modeled by a probability space (Ω, \mathcal{F}, P) . The partial information about the true state evolves exogenously over time, represented by a filtration $(\mathcal{F}_t)_{t \geq 0}$. The player's stopping decisions may of course use this information, so a feasible plan is in principle a stopping time; see Section 2.1 for the formal definition of strategies.

As usual in timing games, we focus only on situations (resp. histories) in which no player has stopped, yet. Therefore, the game ends as soon as any of the players stops. A player who

is the single one to stop first is called the *leader*. In this case the other player becomes the *follower*. Their respective payoffs are determined by the two stochastic processes L and F . Both processes incorporate the possible effect of an (optimal, contingent) stopping decision that the follower might have in a more primitive model, given that the opponent has already stopped – as in Example 2.3 below. If the game ends by both players stopping simultaneously, their payoffs are determined by the third process M . All processes are measured in the same numeraire, say, discounted to time 0, and the players are risk neutral.⁷

Equilibria will obviously be based on solving optimal stopping problems involving the three underlying payoff processes. We need to make some weak regularity assumptions in order to have well defined problems in the following.

Assumption 2.1.

- (i) (Ω, \mathcal{F}, P) is a fixed probability space equipped with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions (i.e., \mathbf{F} is right-continuous and complete).
- (ii) The processes L , F and M are adapted, right-continuous (a.s.) and of class (D), M having an extension with $E[|M_\infty|] < \infty$.
- (iii) $\min(L, F)$ is upper-semi-continuous from the left in expectation, in fact on $[0, \infty]$ if we put $L_\infty = F_\infty = M_\infty$.

Remark 2.2.

- (i) The payoff processes L , F and M do not have to be random; deterministic ones are just a special case. Even then the probability space and filtration might be nontrivial and represent possible public randomization devices. The current payoffs at any time t just should be known by the public information \mathcal{F}_t .
- (ii) Two important general technical issues are measurability, in particular concerning strategies that we address below, and integrability. We need to ensure that expectations are always well defined and that pointwise converging random variables converge in expectation, too. Class (D) is the possibly weakest integrability condition we can work with.⁸ Boundedness would be much too strong for many applications (e.g., involving Brownian motion).

⁷Alternatively, one can interpret the payoff processes as measured in discounted “utils”.

⁸A measurable process X is of class (D) if the family $\{X_\tau : \tau < \infty \text{ a.s. a stopping time}\}$ is uniformly integrable. Then the family is bounded in expectation and pointwise convergence of X at a stopping time implies convergence in $L^1(P)$ as well. This is a mild regularity condition implied, e.g., by either $E[\sup_t |X_t|] < \infty$ or $\sup_\tau E[|X_\tau|^p] < \infty$ for some $p > 1$. We may equivalently define any $X_\infty \in L^1(P)$ and consider *all* stopping times (possibly taking the value ∞) in the previous set; cf. Lemma A.1.

- (iii) It depends on the model whether there is a natural payoff if both players “never stop”, which may be some limit of M or of L . In the latter case we simply set $M_\infty := L_\infty$ and work with M_∞ for a unified payoff notation. For convenience, we also formally define

$$F_\infty := M_\infty.$$

- (iv) In order to have any general existence results for equilibria, some path regularity of the payoff processes is necessary, as can be seen clearly even in the deterministic, single agent case. Nevertheless, it suffices for us to have upper-semi-continuity from the left only in expectation.⁹ In optimal stopping problems this property is also needed for existence. We use it for equilibria in mixed strategies when there is a (local) second-mover advantage. It is of course only required for L if that never exceeds F . Indeed, one could restrict attention to intervals $[\tau, \inf\{t \geq \tau \mid L_t > F_t\}]$, where τ is a stopping time; the area $\{L > F\}$ is only relevant at transitions. The assumption is satisfied, e.g., if the paths of L and F are a.s. (upper-semi-)continuous from the left.¹⁰

Example 2.3. Let us consider a market exit problem as a simple example for a stochastic timing game with second-mover advantage, i.e., $F \geq L$, like in the classical war of attrition.¹¹

Suppose that two firms are operating in one market such that duopoly returns π^D might not be sustainable in the long run, depending on uncertain exogenous conditions. While each firm would in general like the opponent to leave the market in order to earn the monopoly profit $\pi^M \geq \pi^D$, it might be too costly to wait for that possibly random event. Each firm thus decides on times when waiting becomes no longer promising, and at which to leave the market if the other is still present.

The payoff processes are then:

$$\begin{aligned} L_t = M_t &:= \int_0^t \pi_s^D ds, \\ F_t &:= L_t + \operatorname{ess\,sup}_{t \leq \tau^F \in \mathcal{T}} E \left[\int_t^{\tau^F} \pi_s^M ds \mid \mathcal{F}_t \right] \end{aligned} \quad (2.1)$$

for all $t \in [0, \infty]$ (the convention $F_\infty = M_\infty = L_\infty$ here holds naturally). Any monopolist may drop out at a stopping time τ^F , since the monopoly return need not be profitable, either; then immediate exit is the dominant strategy and the second-mover advantage will not be strict. However, we do not model the strategy of a single remaining firm, but incorporate the corresponding optimal decision in the payoff processes. The idea of subgame perfection

⁹Upper-semi-continuity from the left in expectation means $E[L_\tau \wedge F_\tau] \geq \limsup_n E[L_{\tau_n} \wedge F_{\tau_n}]$ for any sequence of stopping times (τ_n) that is a.s. increasing to a stopping time τ .

¹⁰Then $\limsup_{s \nearrow t} (L_s \wedge F_s) \leq (\limsup_{s \nearrow t} L_s) \wedge (\limsup_{s \nearrow t} F_s) \leq L_t \wedge F_t$ for all $t \in [0, \infty]$ a.s., and we note that L and F are of class (D).

¹¹For typical examples of preemption type, see Riedel and Steg (2014).

requires that the latter be chosen optimally.

Assumption 2.1 is satisfied in this example if π^D and π^M are adapted and $P \otimes dt$ -integrable, as all processes are then bounded by an integrable random variable. It follows from our discussion in Section 3.1 below that there exists a right-continuous process F such that the relation (2.1) holds even when substituting t by a general stopping time τ , which is one of the most important results in continuous-time stopping.¹²

2.1 Mixed strategies and equilibrium concept

The concept of subgame-perfect equilibrium for stochastic timing games of Riedel and Steg (2014) is as follows. The feasible decision nodes in continuous time are all *stopping times*. Therefore we consider any stopping time as the beginning of a subgame, with the connotation that no player has stopped before. Let \mathcal{T} denote the set of all stopping times w.r.t. our filtration \mathbf{F} .

We hence specify complete plans of actions for all subgames, taking the form of (random) distribution functions over time. In order to aggregate strategies for the whole game, one requires *time consistency*, meaning that Bayes' law has to be respected wherever it applies.

Additional strategy extensions are needed for subgames with first-mover advantages, to model preemption appropriately in continuous time.¹³ Therefore we use the generalization of the concept of Fudenberg and Tirole (1985) to stochastic models developed in Riedel and Steg (2014) – which preserves the interpretation of discrete-time limits. These extensions are introduced immediately, although one can abstract from them in the discussion of games with a second-mover advantage. We will take them up later for general games.

Definition 2.4. An *extended mixed strategy* for player $i \in \{1, 2\}$ in the subgame starting at $\vartheta \in \mathcal{T}$, also called ϑ -*strategy*, is a pair of processes $(G_i^\vartheta, \alpha_i^\vartheta)$ taking values in $[0, 1]$, respectively, with the following properties.

- (i) G_i^ϑ is adapted. It is right-continuous and nondecreasing with $G_i^\vartheta(t) = 0$ for all $t < \vartheta$, a.s.
- (ii) α_i^ϑ is progressively measurable.¹⁴ It is right-continuous where $\alpha_i^\vartheta < 1$, a.s.¹⁵

¹²This issue is somewhat more delicate for L if we have an example in which the leader's payoff also depends on the stopping time eventually chosen by the follower, as in an entry model. The follower's decision involves of course no optimality condition with respect to the leader's payoff stream, which may make the leader's payoff discontinuous from the right in expectation (which optimality exactly prevents for the follower's payoff). In general that problem will not arise in diffusion models, however.

¹³See also Hendricks and Wilson (1992) on (non-)existence of equilibria in deterministic preemption games.

¹⁴Formally, the mapping $\alpha_i^\vartheta: \Omega \times [0, t] \rightarrow \mathbb{R}$, $(\omega, s) \mapsto \alpha_i^\vartheta(\omega, s)$ must be $\mathcal{F}_t \otimes \mathcal{B}([0, t])$ -measurable for any $t \in \mathbb{R}_+$. It is a stronger condition than adaptedness, but weaker than optionality, which we automatically have for G_i^ϑ by right-continuity. Progressive measurability implies that $\alpha_i^\vartheta(\tau)$ will be \mathcal{F}_τ -measurable for any $\tau \in \mathcal{T}$.

¹⁵This means that with probability 1, $\alpha_i^\vartheta(\cdot)$ is right-continuous at all $t \in [0, \infty)$ for which $\alpha_i^\vartheta(t) < 1$. Since we are here only interested in *symmetric* games, we may demand the extensions $\alpha_i^\vartheta(\cdot)$ to be right-continuous

(iii)

$$\alpha_i^\vartheta(t) > 0 \Rightarrow G_i^\vartheta(t) = 1 \quad \text{for all } t \geq 0, \text{ a.s.}$$

We further define $G_i^\vartheta(0-) \equiv 0$, $G_i^\vartheta(\infty) \equiv 1$ and $\alpha_i^\vartheta(\infty) \equiv 1$ for every extended mixed strategy.

Note that we do not require the players to stop in finite time. Then player i may for instance decide to stop simply at some stopping time $\tau \geq \vartheta$, which is interpreted as a *pure* strategy and corresponds to $G_i^\vartheta(t) = \mathbf{1}_{t \geq \tau}$ for all $t \geq 0$ (and $\alpha_i^\vartheta = \mathbf{1}_{t \geq \infty}$). If $\alpha_i^\vartheta \equiv 0$ on $[0, \infty)$, we loosely speak of a “standard” mixed strategy. Such extended mixed strategies are completely equivalent to mixed strategies in the analogous model without extensions α_i^ϑ .

Definition 2.5. Given two extended mixed strategies $(G_i^\vartheta, \alpha_i^\vartheta)$, $(G_j^\vartheta, \alpha_j^\vartheta)$, $i, j \in \{1, 2\}$, $i \neq j$, the *payoff* of player i in the subgame starting at $\vartheta \in \mathcal{T}$ is

$$\begin{aligned} V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) := & E \left[\int_{[0, \hat{\tau}^\vartheta)} (1 - G_j^\vartheta(s)) L_s dG_i^\vartheta(s) + \int_{[0, \hat{\tau}^\vartheta)} (1 - G_i^\vartheta(s)) F_s dG_j^\vartheta(s) \right. \\ & \left. + \sum_{s \in [0, \hat{\tau}^\vartheta)} \Delta G_i^\vartheta(s) \Delta G_j^\vartheta(s) M_s + \lambda_{L,i}^\vartheta L_{\hat{\tau}^\vartheta} + \lambda_{L,j}^\vartheta F_{\hat{\tau}^\vartheta} + \lambda_M^\vartheta M_{\hat{\tau}^\vartheta} \middle| \mathcal{F}_\vartheta \right]. \end{aligned}$$

At $\hat{\tau}^\vartheta := \inf\{t \geq \vartheta \mid \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0\}$, the extensions α_i^ϑ determine final outcome probabilities $\lambda_{L,i}^\vartheta$, $\lambda_{L,j}^\vartheta$ and λ_M^ϑ . Their definition is given in Appendix C for completeness; it is a simplification of that in Riedel and Steg (2014), thanks to the slightly stronger regularity here. Note that if both players reserve some mass for $t = \infty$ (whence $\hat{\tau}^\vartheta = \infty$, $\lambda_{L,i}^\vartheta = \lambda_{L,j}^\vartheta = 0$), the corresponding payoff will be $(1 - G_i^\vartheta(\infty-))(1 - G_j^\vartheta(\infty-))M_\infty$, since we have defined $G_i^\vartheta(\infty) = \alpha_i^\vartheta(\infty) = 1$.

The pathwise integrals do include possible jumps of the *right-continuous* integrators at 0, since player i can become leader/follower from an initial jump of $G_i^\vartheta/G_j^\vartheta$, respectively. The payoffs are indeed well defined under Assumption 2.1 and bounded in expectation – uniformly across all feasible strategies; cf. Lemma A.2.

To aggregate ϑ -strategies across subgames, time consistency in the form of Bayes’ law has to hold.

Definition 2.6. An *extended mixed strategy* for player $i \in \{1, 2\}$ in the stopping game is a family

$$(G_i, \alpha_i) := (G_i^\vartheta, \alpha_i^\vartheta)_{\vartheta \in \mathcal{T}}$$

of extended mixed strategies for all subgames $\vartheta \in \mathcal{T}$.

also where they take the value 0, which simplifies the definition of outcomes. See Section 3 of Riedel and Steg (2014) for issues with asymmetric games and corresponding weaker regularity restrictions.

An extended mixed strategy (G_i, α_i) is *time-consistent* if for all $\vartheta \leq \vartheta' \in \mathcal{T}$

$$\vartheta' \leq t \in \mathbb{R}_+ \Rightarrow G_i^\vartheta(t) = G_i^\vartheta(\vartheta' -) + (1 - G_i^\vartheta(\vartheta' -))G_i^{\vartheta'}(t) \quad \text{a.s.}$$

and

$$\vartheta' \leq \tau \in \mathcal{T} \Rightarrow \alpha_i^\vartheta(\tau) = \alpha_i^{\vartheta'}(\tau) \quad \text{a.s.}$$

Note that time consistency implies in particular that for any two subgames $\vartheta, \vartheta' \in \mathcal{T}$ we must have $G_i^\vartheta \equiv G_i^{\vartheta'}$ (a.s.) on the event $\{\vartheta = \vartheta'\}$, as one would reasonably expect. The equilibrium concept is then natural.

Definition 2.7. A *subgame-perfect equilibrium* for the timing game is a pair $(G_1, \alpha_1), (G_2, \alpha_2)$ of time-consistent extended mixed strategies such that for all $\vartheta \in \mathcal{T}$, $i, j \in \{1, 2\}$, $i \neq j$, and extended mixed strategies $(G_a^\vartheta, \alpha_a^\vartheta)$

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \geq V_i^\vartheta(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \quad \text{a.s.},$$

i.e., such that every pair $(G_1^\vartheta, \alpha_1^\vartheta), (G_2^\vartheta, \alpha_2^\vartheta)$ is an *equilibrium* in the subgame at $\vartheta \in \mathcal{T}$, respectively.

3 Best replies and optimal stopping

The payoffs in Definition 2.5 are apparently linear in strategies. In this section we derive a more explicit representation of this linearity, which will be very helpful for rigorous proofs to verify equilibria, but also for necessity arguments. To determine or verify any best replies, one needs to maximize over (extended) mixed strategies against these same objects in general, of course. Here we make related statements such as “any stopping time in the support of the mixed strategy needs to be optimal” precise. We further introduce the central concepts from the theory of optimal stopping, notably the *Snell envelope*, which plays a crucial role in the following representation and interpretation of mixed strategies in equilibrium.

The following arguments concern the distributions G_i^ϑ , so we neglect the extensions until Section 7 for simplicity (which formally means restricting to $\alpha_i^\vartheta \equiv \mathbf{1}_{t \geq \infty}$, as mentioned in Section 2.1, i.e., to “standard” mixed strategies).

Now, for the alternative representation of the payoff of player $i \in \{1, 2\}$ in the subgame at $\vartheta \in \mathcal{T}$, we introduce the process S_i^ϑ given by

$$S_i^\vartheta(t) := \int_{[0, t)} F_s dG_j^\vartheta(s) + \Delta G_j^\vartheta(t) M_t + (1 - G_j^\vartheta(t)) L_t \quad (3.1)$$

for all $t \in [0, \infty)$, where G_j^ϑ is a given feasible mixed strategy for the opponent $j \in \{1, 2\} \setminus i$.

Lemma A.3 shows that this process is well behaved: it is *optional*¹⁶ and of class (D). With $M_\infty \in L^1(P)$, we can extend the definition of S_i^ϑ in (3.1) to $t = \infty$ implying also $S_i^\vartheta(\infty) \in L^1(P)$.¹⁷

Thanks to Lemma A.2 and $S_i^\vartheta(\infty) \in L^1(P)$, we can integrate S_i^ϑ by any feasible dG_i^ϑ to see that the expected payoff of player i in the subgame beginning at $\vartheta \in \mathcal{T}$ can be written as¹⁸

$$V_i^\vartheta(G_i^\vartheta, G_j^\vartheta) = E \left[\int_{[0, \infty]} S_i^\vartheta(t) dG_i^\vartheta(t) \middle| \mathcal{F}_\vartheta \right]. \quad (3.2)$$

As is to be expected by the linearity of (3.2), there exists a best reply only if there is one that is a pure strategy.

Lemma 3.1. *Fix $\vartheta \in \mathcal{T}$ and let G_i^ϑ and G_j^ϑ be feasible. Then*

$$V_i^\vartheta(G_i^\vartheta, G_j^\vartheta) \leq \text{ess sup}_{\vartheta \leq \tau \in \mathcal{T}} E[S_i^\vartheta(\tau) | \mathcal{F}_\vartheta] \quad a.s.$$

Proof: In Appendix B.1.

From the lemma (and more explicitly from its proof) we see that a feasible strategy G_i^ϑ will be a best reply to G_j^ϑ if and only if for any stopping time τ^* such that $dG_i^\vartheta(\tau^*) > 0$,¹⁹

$$E[S_i^\vartheta(\tau^*) | \mathcal{F}_\vartheta] \geq E[S_i^\vartheta(\tau) | \mathcal{F}_\vartheta] \quad \forall \vartheta \leq \tau \in \mathcal{T}$$

and we generally have to solve the optimal stopping problem on the right in Lemma 3.1.

The central aspect of continuous-time games of timing is their inherent discontinuity, even if the underlying data (here L , F , and M) is continuous. For instance, from the definition of S_i^ϑ in (3.1) it is now immediately clear that a best reply cannot put any joint mass points where $F > M$, since $\lim_{\varepsilon \searrow 0} S_i^\vartheta(t + \varepsilon) - S_i^\vartheta(t) = \Delta G_j^\vartheta(t)(F_t - M_t)$ by right-continuity of L ; this will be a frequent argument. Depending on G_j^ϑ , there need not exist any stopping time that actually attains the value of the problem, as S_i^ϑ may have various kinds of discontinuities. Dealing with such discontinuities will be one of our major issues.

In the following subsection we present some crucial facts from the general theory of optimal

¹⁶This means that S_i^ϑ is measurable w.r.t. the optional σ -field on the product space $\Omega \times \mathbb{R}_+$, which is generated by all right-continuous adapted processes, or equivalently by the random intervals $[0, \tau)$, $\tau \in \mathcal{T}$.

¹⁷Note that the integral in (3.1) converges, as it is bounded by $\int_{[0, \infty)} |F_s| dG_j^\vartheta(s) \in L^1(P)$ thanks to Lemma A.2.

¹⁸An application of Fubini's theorem using footnote 17 yields in particular

$$\begin{aligned} \int_{[0, \infty)} (1 - G_i^\vartheta(s)) F_s dG_j^\vartheta(s) &= \int_{[0, \infty)} \int_{[0, \infty]} \mathbf{1}_{t > s} dG_i^\vartheta(t) F_s dG_j^\vartheta(s) \\ &= \int_{[0, \infty)} \int_{[0, \infty)} \mathbf{1}_{s < t} F_s dG_j^\vartheta(s) dG_i^\vartheta(t) = \int_{[0, \infty)} \int_{[0, t)} F_s dG_j^\vartheta(s) dG_i^\vartheta(t) \in L^1(P). \end{aligned}$$

¹⁹This means that $G_i^\vartheta(t) > G_i^\vartheta(\tau -)$ for all $t > \tau$ a.s.

stopping in continuous time, providing in particular sufficient (and basically necessary) conditions for the existence of optimal stopping times and their characterization in terms of the Snell envelope. The latter is in fact our main tool to derive and represent mixed equilibrium strategies.

3.1 Optimal stopping in continuous time

As a motivating stopping problem to present the theory, consider the unilateral problem of when to become the leader optimally, i.e., supposing the opponent will never act. This problem will play an important role in the following.²⁰

It is well established how to characterize the solution of the optimal stopping problem

$$V_L(0) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}} E[L_\tau]$$

given Assumption 2.1. In fact, our payoff process L is right-continuous (hence optional) and of class (D), such that we can apply the general theory of optimal stopping as in, e.g., Mertens (1972) and Bismut and Skalli (1977): There exists a smallest supermartingale U_L dominating the payoff process L , called the *Snell envelope* of L , which satisfies

$$U_L(\vartheta) = \operatorname{ess\,sup}_{\vartheta \leq \tau \in \mathcal{T}} E[L_\tau | \mathcal{F}_\vartheta] \quad \text{a.s.} \quad (3.3)$$

for all stopping times $\vartheta \in \mathcal{T}$. In particular $U_L(0) = V_L(0)$. We remark that one can very well define the RHS of (3.3) for any $\vartheta \in \mathcal{T}$, but the key insight is that there exists a well behaved *process* $U_L = (U_L(t))_{t \geq 0}$, which one can evaluate at any stopping time ϑ to know the continuation value there. In view of the dynamic programming principle, we do need to consider continuation problems at stopping times; the latter are feasible quantities, but much richer than deterministic times.

Now U_L is optional and of class (D) as well²¹ and such supermartingales have very convenient regularity properties: There exists a Doob-Meyer decomposition²²

$$U_L = M_L - D_L$$

that we extensively use, with a uniformly integrable, right-continuous martingale²³ M_L and a nondecreasing, predictable and integrable process D_L . The latter can be interpreted as measuring the *expected loss from stopping too late*: If we postpone any stopping to $\tau \geq 0$,

²⁰To stay in the framework of the game, for finding a (pure) best reply to $G_j^0 = \mathbf{1}_{t \geq \infty}$ we have to use the payoff M_∞ for not stopping in finite time. Recall our convention of setting $L_\infty = M_\infty$, however.

²¹See Mertens (1972), Théorème T4 for the existence and Théorème T5 and proof for U_L being of class (D).

²²See Mertens (1972), Théorème T3.

²³Therefore, the crucial optional sampling holds: $M_L(\sigma) = E[M_L(\tau) | \mathcal{F}_\sigma]$ for all $\sigma \leq \tau \in \mathcal{T}$. Further, M_L has a last element $M_L(\infty)$ to which it converges in $L^1(P)$.

then we cannot achieve more than $E[U_L(\tau)] = U_L(0) - E[D_L(\tau)]$, even if we stop optimally from τ onwards.

Reflecting the dynamic programming principle, the value process U_L is a martingale as long as there still exists a future time τ giving at least the same value in expectation as stopping immediately. Whether there exists any *optimal* stopping time depends on the continuity properties of D_L . If L is upper-semi-continuous in expectation (as by Assumption 2.1 (iii) if $L \leq F$, e.g.), D_L has left-continuous paths a.s.²⁴ By right-continuity of L , D_L will be even continuous.²⁵ With D_L left-continuous, there exist the optimal stopping times²⁶

$$\tau_L(\vartheta) := \inf\{t \geq \vartheta \mid U_L(t) = L_t\} \quad \text{and} \quad \tau^L(\vartheta) := \inf\{t \geq \vartheta \mid D_L(t) > D_L(\vartheta-)\}. \quad (3.4)$$

They are the respectively smallest and largest stopping times after $\vartheta \in \mathcal{T}$ attaining²⁷

$$U_L(\vartheta) = E[L_{\tau_L(\vartheta)} \mid \mathcal{F}_\vartheta] = E[L_{\tau^L(\vartheta)} \mid \mathcal{F}_\vartheta] \quad \text{a.s.} \quad (3.5)$$

Hence, by optimality it must hold that $U_L = L$ a.s. at any point of increase of D_L , which implies²⁸

$$\int_0^\infty (U_L(t) - L_t) dD_L(t) = 0 \quad \text{a.s.} \quad (3.6)$$

4 Equilibria in pure strategies

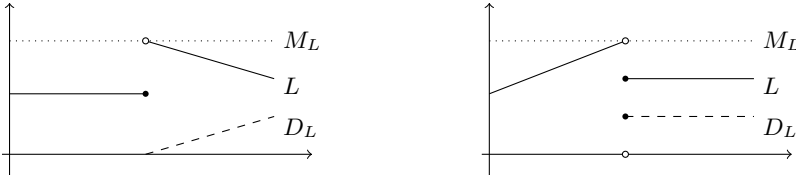
In symmetric games with a systematic second-mover advantage $F \geq L$, it is straightforward to identify certain subgame-perfect equilibria in pure strategies. Player j , say, just has to stop sufficiently late, such that i will solve the problem of optimally stopping L presented in Section 3.1. We show in this section that such pure strategy equilibria typically entail asymmetric

²⁴See Bismut and Skalli (1977), Théorème II.2 and proof. (Semi-) Continuity in expectation is in general weaker than the corresponding path property from the left.

Our payoff processes are not necessarily positive. However, if L is optional and of class (D), the same will be true for its negative part $L^- := \max(-L, 0)$, which thus has a Snell envelope $U_{L^-} = M_{L^-} - D_{L^-}$ decomposing into a uniformly integrable right-continuous martingale M_{L^-} and an integrable increasing process D_{L^-} . Then $M_{L^-} - L^- \geq 0$, implying $L + M_{L^-} \geq 0$. Adding the martingale M_{L^-} neither affects L being optional, of class (D), or (semi-) continuous in expectation, nor any optimal stopping times for L .

²⁵See Bismut and Skalli (1977), (2.15), where right-continuity of the payoff process implies in fact $Z^+ = X$.

²⁶*Example:* L not upper-semi-continuous $\Rightarrow \inf\{D_L > 0\}$ not optimal.



²⁷See Bismut and Skalli (1977), Théorème II.3.

²⁸If D_L is continuous, U_L inherits right-continuity from M_L . Then, by (3.4), (3.5) and right-continuity of $U_L - L$, $\inf\{t \in \mathbb{R}_+ \mid \int_0^t \mathbf{1}_{U_L - L \geq \varepsilon} dD_L > 0\} = \infty$ a.s. for any $\varepsilon > 0$, i.e., $U_L - L < \varepsilon dD_L$ -a.e. with probability one, implying the claim. (3.6) also holds without right-continuity of $U_L - L$, if L is upper-semi-continuous in expectation; see Remark B.1 in the appendix.

payoffs, however. The respective roles of the players have to be determined before the game starts, and correspondingly who obtains the higher payoff. With mixed strategies that we will consider thereafter, one can obtain equilibria with symmetric payoffs that do not create another strategic conflict outside the model.

Stopping sufficiently late to support a pure strategy equilibrium need not be “never” as in the previous section; whenever it is optimal to stop L , it must not be worthwhile for i to wait until j stops, in order to become follower then. This will be the case, e.g., if j stops only at times where $F = L$ – or simply at ∞ .

The easiest example is thus $G_j^\vartheta = \mathbf{1}_{t=\infty}$ and $G_i^\vartheta = \mathbf{1}_{t \geq \tau_L(\vartheta)}$ for all $\vartheta \in \mathcal{T}$, or analogously with $\tau^L(\cdot)$ defined in (3.4). In either case waiting is indeed optimal for j on $[0, \infty)$, because there are expected gains of L on any interval $[\vartheta, \tau^L(\vartheta)]$, and F dominates L at both $\tau_L(\vartheta)$ and $\tau^L(\vartheta)$.

There can also be quite complex patterns based on the same logic, but with players switching roles across subgames. This can be illustrated best with a little more structure as in Example 2.3, where the follower’s optimal stopping times will be “sufficiently late” for an equilibrium. However, the arguments generalize a bit: the exploited properties are that $F \geq L \geq M$ and that F is a supermartingale, i.e., that one becomes follower the sooner the better. Then stopping before $\tau_L(\vartheta)$ is dominated (for any G_j^ϑ).

Lemma 4.1. *Suppose $F \geq L \geq M$ and that F is a supermartingale. Fix $\vartheta \in \mathcal{T}$. For any feasible G_j^ϑ and stopping time $\tau_i \geq \vartheta$,*

$$E\left[S_i^\vartheta((\tau_i \vee \tau_L(\vartheta))+) \mid \mathcal{F}_\vartheta\right] \geq E\left[S_i^\vartheta(\tau_i) \mid \mathcal{F}_\vartheta\right]$$

and

$$E\left[S_i^\vartheta(\tau_L(\vartheta)+) \mid \mathcal{F}_\vartheta\right] \geq E\left[L_{\tau_L(\vartheta)} \mid \mathcal{F}_\vartheta\right], \quad a.s.$$

Both inequalities also hold with $\tau^L(\vartheta)$.

Proof: In Appendix B.1.

Note that the supermartingale property of F is important for the result, to ensure relatively high payoffs in case one becomes follower before the optimum of L is reached. It is not sufficient that there are even strictly better future stopping times for L and that $F \geq L$: If G_j^ϑ puts mass between ϑ and $\tau_L(\vartheta)$ where F still dominates L , but where both are very low, then it may be worthwhile to secure the current payoff L_ϑ due to the risk of becoming follower while waiting for the optimum of L . An alternative condition would be that L is a *submartingale* on $[\vartheta, \tau_L(\vartheta)]$.

In Example 2.3, $L = \int_0^\cdot \pi^D ds$ is the duopolists’ payoff process. Then the optimal stopping times in the follower’s problem are sufficiently late to support an equilibrium: the perspective

to become follower (monopolist) at a time when immediate exit is optimal has no value, and it leads to ceding when π^D seems an unsustainable loss – at $\tau_L(\vartheta)$. Indeed, as a monopolist stops $\int_0^\cdot \pi^M ds$ with $\pi^M \geq \pi^D$, that optimal stopping time satisfies $\tau_F(\vartheta) \geq \tau_L(\vartheta)$. Furthermore, it holds that $F = L = M$ a.s. at $\tau_F(\vartheta)$, so in particular simultaneous stopping is feasible on $\{\tau_L(\vartheta) = \tau_F(\vartheta)\}$ by $F = M$. These properties generate a whole class of equilibria with varying roles of the players, decided by events C at $\tau_L(\vartheta)$.

Proposition 4.2. *Suppose $F \geq L \geq M$ and that F is a supermartingale. Fix $\vartheta \in \mathcal{T}$ and consider a stopping time $\tau_F(\vartheta) \geq \tau_L(\vartheta)$ a.s., such that at $\tau_F(\vartheta)$ we have $F = L$, and more specifically $F = M$ on the subset $\{\tau_F(\vartheta) = \tau_L(\vartheta)\}$ a.s. – e.g., $\tau_F(\vartheta) := \inf\{t \geq \vartheta \mid F_t = M_t\}$. Then, for any given event $C \in \mathcal{F}_{\tau_L(\vartheta)}$, the pure strategies corresponding to*

$$\tau_1^* = \tau_L(\vartheta)\mathbf{1}_C + \tau_F(\vartheta)\mathbf{1}_{C^c} \quad \text{and} \quad \tau_2^* = \tau_L(\vartheta)\mathbf{1}_{C^c} + \tau_F(\vartheta)\mathbf{1}_C$$

form an equilibrium in the subgame beginning at ϑ .

Proof: In Appendix B.1.

Equilibria in pure strategies typically involve asymmetric payoffs, for instance if $F > L$ at $\tau_L(\vartheta)$ in those that we have specified. Consequently, there arises a coordination problem before the start of the game, each player wanting to become follower eventually. This problem is even aggravated in the equilibria of Proposition 4.2, where the roles may switch across subgames. For this reason such equilibria are also not easy to aggregate to a subgame-perfect equilibrium: for each subgame starting at some $\vartheta \in \mathcal{T}$, an event $C \in \mathcal{F}_{\tau_L(\vartheta)}$ has to be agreed on that determines the respective roles.

Maybe even more importantly, no player can obtain the preferred follower payoff by taking or threatening to take a certain action, but only by the threat of taking *no* action for a longer time, which has to induce the opponent to stop. Effectively, players compete in the credibility to take no action. Such problems can be avoided by allowing for mixed strategies, making the players indifferent about the roles when stopping occurs. This is our topic in the following.

5 Equilibria in mixed strategies

The very universal principle of the Snell envelope allows us to construct equilibria in mixed strategies in our general setting as well. But we do not only get existence: our equilibrium strategies can be clearly interpreted like the Snell envelope itself. Recall that the compensator relates to the expected loss from stopping too late.

The logic of the following equilibria in symmetric games is this. If $F \geq L$ but without the other conditions of Lemma 4.1, waiting for future optimal times to stop L may not be a *dominant* strategy. Nevertheless, the players have an incentive and the possibility to

coordinate on not stopping too early, which can be extended until the latest optimal time $\tau^L(\vartheta)$ to stop L . To cross that point, however, any player has to be compensated by some chance to become follower with $F > L$, because otherwise any delay would definitely be costly. Of course the opponent has to be willing to provide that chance, so we identify the appropriate rate to compensate *exactly* the impending loss $dD_L > 0$, to make both indifferent.

This principle does not work where $L > F$, however, when there would be much more intense stopping due to a preemption incentive (see Section 7). On the other hand, even if we were considering only games with a global second-mover advantage, there may be equilibria with even higher (symmetric) payoffs – if simultaneous stopping is feasible and sufficiently profitable at some future point, precisely where $M \geq F > L$. For these reasons we need to adapt the appropriate payoff process that players can coordinate on.

Theorem 5.1. *Consider a subgame beginning at $\vartheta \in \mathcal{T}$ with $F_\vartheta \geq L_\vartheta$ a.s. on $\{\vartheta < \infty\}$, and another stopping time $\tau^\vartheta \in \mathcal{T}$ taking values in $[\vartheta, \inf\{t \geq \vartheta \mid F_t < L_t\}]$ a.s. Define the payoff process $\tilde{L}^{\tau^\vartheta} := \mathbf{1}_{t < \tau^\vartheta} L + \mathbf{1}_{t \geq \tau^\vartheta} \max(F_{\tau^\vartheta}, M_{\tau^\vartheta})$ and as $D_L^{\tau^\vartheta}$ the compensator of its Snell envelope.*

Then there exists a payoff-symmetric equilibrium with mixed strategies satisfying

$$G_i^\vartheta(t) = 1 - \mathbf{1}_{t < \tau^\vartheta} \exp\left(-\int_\vartheta^t \frac{dD_L^{\tau^\vartheta}(s)}{F_s - L_s}\right) \quad (5.1)$$

and

$$G_j^\vartheta(t) = 1 - \mathbf{1}_{t < \tau^\vartheta} \exp\left(-\int_\vartheta^t \frac{\mathbf{1}_{F_s > L_s} dD_L^{\tau^\vartheta}(s)}{F_s - L_s}\right), \quad (5.2)$$

$i, j \in \{1, 2\}$, $i \neq j$, iff $\Delta G_i^\vartheta(M - F) \leq 0$ at $\inf\{t \in \mathbb{R}_+ \mid G_i^\vartheta(t) = 1\} < \tau^\vartheta$ and $\Delta G_i^\vartheta(M - F) \geq 0$ at $\tau^\vartheta < \infty$ a.s.

There further exists a symmetric equilibrium with both players using the strategy (5.1) iff $\Delta G_i^\vartheta(M - F) = 0$ a.s. at $\inf\{t \in \mathbb{R}_+ \mid G_i^\vartheta(t) = 1\} < \tau^\vartheta$.

Proof: In Appendix B.1.

The “endpoint condition” at $\inf\{t \in \mathbb{R}_+ \mid G_i^\vartheta(t) = 1\} =: \tau_i^{G, \vartheta}(1)$ seems contradictory, but it plays the following role. First suppose $\Delta G_i^\vartheta(\tau^\vartheta) > 0$, so $\tau_i^{G, \vartheta}(1) = \tau^\vartheta$ and there is a joint terminal jump. This means that the players coordinate on the terminal payoff M_{τ^ϑ} , which is only feasible if $M_{\tau^\vartheta} \geq F_{\tau^\vartheta}$ (recall $M_\infty = F_\infty$ by convention).

G_i^ϑ can indeed jump to 1 before τ^ϑ , where $F = L$. There we choose G_j^ϑ continuous to address the case $F = L > M$, where payoffs are hence symmetric. This choice can only be an equilibrium, however, if indeed $F \geq M$. Otherwise j could obtain a higher value by stopping at $\tau_i^{G, \vartheta}(1)$ and not supporting the equilibrium earlier on; we could then simply adjust τ^ϑ to ensure the correct continuation values.

Finally, G_i^ϑ can reach 1 also continuously (then $G_j^\vartheta \equiv G_i^\vartheta$). This case is one reason why we write “ $\max(F_{\tau^\vartheta}, M_{\tau^\vartheta})$ ” in the definition of $\tilde{L}^{\tau^\vartheta}$, although we require $\Delta G_i^\vartheta(M - F) \geq 0$ at τ^ϑ . Even if $\Delta G_i^\vartheta = 0$, the terminal value of $\tilde{L}^{\tau^\vartheta}$ determines the continuation values on which players coordinate earlier on. Putting M_{τ^ϑ} regardless would not be correct. Another reason is that we will indeed obtain continuation equilibria with payoff $\max(F_{\tau^\vartheta}, M_{\tau^\vartheta})$ when we take up *extended* mixed strategies in Section 7.

The strategies here are continuous except for terminal jumps. As motivated above and given the appropriate payoff process $\tilde{L}^{\tau^\vartheta}$, the opponent’s stopping rate $dD_L^{\tau^\vartheta}/(F - L)$ makes each player indifferent when it would seem optimal to secure L beforehand. That probability to obtain $F > L$ exactly compensates any expected loss from forgoing L . The resulting equilibrium payoffs are

$$V_1^\vartheta = V_2^\vartheta = \operatorname{ess\,sup}_{\vartheta \leq \tau \in \mathcal{T}} E \left[\mathbf{1}_{\tau < \tau^\vartheta} L_\tau + \mathbf{1}_{\tau \geq \tau^\vartheta} \max(F_{\tau^\vartheta}, M_{\tau^\vartheta}) \middle| \mathcal{F}_\vartheta \right] := U_L^{\tau^\vartheta}(\vartheta).$$

The proof of Theorem 5.1 is based on martingale arguments. An important aspect is to take care of the different kinds of jumps in the strategies and to ensure that the underlying payoff process $\tilde{L}^{\tau^\vartheta}$ has the necessary properties (e.g., that $D_L^{\tau^\vartheta}$ is continuous). Where stopping happens continuously, it need not have a rate with respect to time dt , however (though it does in the explicit Brownian example in Section 6); it might only take place on a set of time points of measure 0.²⁹

The strategies are trivial in the present equilibria – given that the endpoint is feasible – if either L or F is a (sub-)martingale on $[\vartheta, \tau^\vartheta]$; then there is no loss from waiting and $dD_L^{\tau^\vartheta} \equiv 0$.³⁰

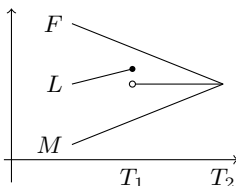
Remark 5.2. It may happen that D_L and hence G_i^ϑ has jumps if L is only upper-semi-continuous from the right (and the left). In the specified equilibria, waiting is always at least as good as obtaining L and there must be indifference at increases of D_L . This is of course not possible with a joint mass point where $L > M$.³¹ Theorem 5.1 remains true if instead $L \equiv M$ (e.g., in an attrition model); see Remark B.1.

The equilibria of Theorem 5.1 can deal so far only with subgames satisfying $F_\vartheta \geq L_\vartheta$. Therefore, if we want to aggregate them to a subgame-perfect equilibrium, we have to assume

²⁹In this case G_i^ϑ would be a *singular* measure, which often appear in optimal control of Brownian models.

³⁰Cf. Riedel and Steg (2014), Theorem 3.3, for this case, but also their Section 4.3 concerning issues in asymmetric games.

³¹*Example:* No symmetric payoff equilibrium if $L (> M)$ not right-continuous.



Waiting is strictly optimal for i at $t \in (T_1, T_2)$ if $G_j(T_2-) > G_j(t)$, hence $G_i(T_2-) = G_i(T_1)$ and $G_j(T_2-) = G_j(T_1)$ by payoff symmetry, giving a continuation payoff $L(T_2)$ on $(T_1, T_2]$. The only symmetric continuation payoff at T_1 is then in $(L(T_2), L(T_1))$ from $\Delta G_i(T_1) = \Delta G_j(T_1) \in (0, 1)$. Waiting is also strictly dominant on $[0, T_1)$, but stopping short of T_1 now yields a higher payoff than stopping at T_1 .

$F \geq L$ for now. Then existence is trivial by setting $\tau^\vartheta = \inf\{t \geq \vartheta \mid M_t > F_t\}$ for all $\vartheta \in \mathcal{T}$. If also $F \geq M$ throughout, we get the further simplification that we can set $\tau^\vartheta \equiv \infty$ and hence $\tilde{L}^{\tau^\vartheta} = L$, $D_L^{\tau^\vartheta} = D_L$. In the latter case the stopping rates do not depend on ϑ , which ensures time consistency.

In general, however, we do not preclude the possibility that at some τ^ϑ , $\max(F, M) > L$ or $\max(F, M) < U_L$, such that $D_L^{\tau^\vartheta} \neq D_L$ on $[\vartheta, \tau^\vartheta]$. Then time consistency requires that τ^ϑ will not be changed if it has not been passed yet: for any two subgames $\vartheta, \vartheta' \in \mathcal{T}$ we should have $\tau^\vartheta = \tau^{\vartheta'}$ on $\{\vartheta \leq \vartheta' \leq \tau^\vartheta\}$ and vice versa (in summary, $\tau^\vartheta = \tau^{\vartheta'}$ on $\{(\vartheta \vee \vartheta') \leq (\tau^\vartheta \wedge \tau^{\vartheta'})\}$, noting that $\vartheta \leq \tau^\vartheta$ and $\vartheta' \leq \tau^{\vartheta'}$ a.s.).

We now have the following payoff-symmetric subgame-perfect equilibria with “standard” mixed strategies for games with systematic second-mover advantage.

Theorem 5.3. *Assume $F \geq L$ and fix $i, j \in \{1, 2\}$, $i \neq j$. If we have an equilibrium as in Theorem 5.1 for any $\vartheta, \vartheta' \in \mathcal{T}$ with $\tau^\vartheta = \tau^{\vartheta'}$ a.s. on $\{(\vartheta \vee \vartheta') \leq (\tau^\vartheta \wedge \tau^{\vartheta'})\}$ – as we do by setting $\tau^\vartheta = \inf\{t \geq \vartheta \mid M_t > F_t\} \forall \vartheta \in \mathcal{T}$, e.g. – then the strategies $(G_1^\vartheta)_{\vartheta \in \mathcal{T}}$ and $(G_2^\vartheta)_{\vartheta \in \mathcal{T}}$ form indeed a subgame-perfect equilibrium.*

Proof: In Appendix B.1.

Even if the time-consistency condition for the family $\{\tau^\vartheta : \vartheta \in \mathcal{T}\}$ postulated in the theorem statement holds, we then have a family $\{D_L^{\tau^\vartheta} : \vartheta \in \mathcal{T}\}$ that needs to induce time-consistent stopping rates $(dG_i^\vartheta)_{\vartheta \in \mathcal{T}}$. This is the main point of Theorem 5.3, given optimality by Theorem 5.1.

6 Example: Exit from a duopoly

In this section we illustrate the simplification we get with games having a systematic second-mover advantage, which we pointed out in the context of Theorem 5.1. Specifically, we determine subgame-perfect equilibrium strategies by explicitly deriving the Snell envelope U_L and its compensator D_L for a version of the market exit game in Example 2.3. The stopping rate during attrition is then represented in terms of a sustained flow of losses from unprofitable operations.

To specify the model, assume that at each time t , discounted duopoly profits are given by

$$\pi_t^D = e^{-rt}(Y_t - c),$$

where $c > 0$ is a constant operating cost and revenues $(Y_t)_{t \geq 0}$ follow a geometric Brownian motion solving $dY = \mu Y dt + \sigma Y dB$. If either firm becomes monopolist, the profit stream changes to

$$\pi_t^M = e^{-rt}(mY_t - c),$$

where $m > 1$. Each firm can decide to leave the market with accumulated payoff $L_t = M_t = \int_0^t e^{-rs}(Y_s - c) ds$, for example if Y gets so low that the revenue does not cover the production costs. In such a phase the game is a war of attrition if monopoly seems profitable. However, it may also be optimal to stop immediately in the follower's problem

$$F_t = L_t + \operatorname{ess\,sup}_{t \leq \tau^F \in \mathcal{T}} E \left[\int_t^{\tau^F} e^{-rs}(mY_s - c) ds \middle| \mathcal{F}_t \right].$$

The latter problem is a standard exercise under the condition $r > \max(\mu, 0)$,³² and its unique solution is to stop as soon as Y falls below the threshold

$$y_m = \frac{\beta_2}{\beta_2 - 1} \frac{r - \mu}{r} \frac{c}{m} < \frac{c}{m},$$

where β_2 is the negative root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$. The value of the stopping problem can be explicitly expressed as

$$F_t - L_t = e^{-rt} \mathbf{1}_{Y_t > y_m} \left[\frac{mY_t}{r - \mu} - \frac{c}{r} - \left(\frac{Y_t}{y_m} \right)^{\beta_2} \left(\frac{my_m}{r - \mu} - \frac{c}{r} \right) \right], \quad (6.1)$$

which shows that F is in fact a continuous process and $F = L (= M) \Leftrightarrow Y \leq y_m$. Hence, for any equilibrium as in Theorem 5.1, we need $\tau^\vartheta \geq \inf\{t \geq \vartheta \mid Y_t \leq y_m\}$ for the endpoint condition. On the other hand, stopping is strictly dominant for a monopolist as soon as $Y \leq y_m$, and so it is in duopoly, where revenues can never exceed those in monopoly. Therefore we can choose $\tau^\vartheta = \inf\{t \geq \vartheta \mid Y_t \leq y_m\}$ without loss and it will lead to a symmetric equilibrium at any $\vartheta \in \mathcal{T}$ as follows.

Since F is a supermartingale for $m \geq 1$ and dominates L , it also dominates the Snell envelope U_L of the latter, such that we have $F = L \Rightarrow F = U_L = L$. Consequently, $\tilde{L}^{\tau^\vartheta}$ in Theorem 5.1 here is just L stopped at τ^ϑ , and the Snell envelope $U_L^{\tau^\vartheta}$ coincides with U_L until τ^ϑ . Applying the RHS of (6.1) with $m = 1$ yields the solution to optimally stopping the leader (duopoly) payoff:

$$U_L(t) = \operatorname{ess\,sup}_{t \leq \tau \in \mathcal{T}} E[L_\tau \mid \mathcal{F}_t] = L_t + e^{-rt} \mathbf{1}_{Y_t > y_1} \left[\frac{Y_t}{r - \mu} - \frac{c}{r} - \left(\frac{Y_t}{y_1} \right)^{\beta_2} \left(\frac{y_1}{r - \mu} - \frac{c}{r} \right) \right].$$

Applying Itô's lemma shows that the monotone part of the supermartingale U_L is just the drift

$$dD_L = -\mathbf{1}_{Y_t < y_1} dL = \mathbf{1}_{Y_t < y_1} e^{-rt}(c - Y_t) dt$$

where immediately stopping L is optimal. With $\tau^\vartheta = \inf\{t \geq \vartheta \mid Y_t \leq y_m\}$ for every $\vartheta \in \mathcal{T}$,

³²This is also necessary and sufficient for the processes to be of class (D) in accordance with Assumption 2.1. Then $-c/r \leq L_t \leq \int_0^\infty e^{-rs} |Y_s - c| ds \in L^1(P)$ and similarly for F , inserting m .

$dD_L^{\vartheta} = dD_L$ and (6.1) we now have a fully explicit symmetric subgame-perfect equilibrium, with payoffs $V_i^{\vartheta}(G_i^{\vartheta}, G_j^{\vartheta}) = U_L(\vartheta)$, respectively.

As $y_m < y_1 < c$, we see that dD_L is simply the stream of losses resulting from unprofitable operations. If a duopolist never hoped to become monopolist, these losses would be too large to keep operating. Here, whenever $Y \in (y_m, y_1)$, both firms are leaving duopoly at a rate that depends directly on those running losses; it is decreasing in Y . The state may rise next to the region (y_1, c) . Then there are still running losses, but the firms suspend mixing because the option to wait for a market recovery is sufficiently valuable. Thus there is no need for a compensation. There will typically be alternating periods of continuous and no mixing. If the state drops to $[0, y_m]$, however, the option to wait for market recovery would be worthless in the face of running losses even if a firm was (sure to become) monopolist, and both firms quit immediately.

7 Equilibria for general symmetric games

7.1 Preemption with extended mixed strategies

In a preemption situation, i.e., when there is a first-mover advantage $L > F$, there typically exist no equilibria in pure strategies in continuous time. Fudenberg and Tirole (1985) and Hendricks and Wilson (1992) show that this issue arises when there is an incentive to wait (L is increasing). If the model is sufficiently regular and the first-mover advantage is strict, then one may have equilibria in (standard) mixed strategies, with one player stopping immediately and the other stopping at a sufficient rate, such that the first would not be able to realize the increase in L . The payoffs are then asymmetric, L and F . However, these equilibria cannot be extended to the boundary of the preemption region; if $L = F$, the necessary stopping rate to support any equilibrium explodes. This observation does not depend on any regularity conditions, the payoff processes can be arbitrarily smooth and deterministic.

Therefore, if we want to allow for any equilibria where preemption will be set off (or also symmetric payoff equilibria where $L > F$), we need to enrich the strategies. The key is to facilitate some partial coordination when players try to stop at the same time, but when simultaneous stopping would be the worst outcome. Hence we make use of the strategy extensions α_i^{ϑ} from Definition 2.4, which have been introduced in Riedel and Steg (2014) and which follow in spirit those of Fudenberg and Tirole (1985). With these extended strategies one can capture the continuous-time limits of symmetric, mixed discrete-time equilibria, which do not suffer from such problems.³³

³³In discrete time, there can be equilibria with a positive probability of simultaneous stopping even if that is the worst outcome, because the players can only assign positive probabilities to the single periods; one cannot circumvent coordination failure by stopping ε after a mass point of the other.

We then obtain the following equilibria of immediate stopping for subgames with a first-mover advantage – here for a symmetric game:³⁴

Proposition 7.1 (Riedel and Steg (2014), Proposition 3.1). *Fix $\vartheta \in \mathcal{T}$ and suppose $\vartheta = \inf\{t \geq \vartheta \mid L_t > F_t\}$ a.s. Then $(G_1^\vartheta, \alpha_1^\vartheta)$, $(G_2^\vartheta, \alpha_2^\vartheta)$ defined by*

$$\alpha_i^\vartheta(t) = \begin{cases} 1 & \text{if } M_t \geq F_t \text{ and } t = \inf\{u \geq t \mid L_u > F_u\}, \\ \mathbf{1}_{L_t > F_t} \frac{L_t - F_t}{L_t - M_t} & \text{else} \end{cases}$$

for any $t \in [\vartheta, \infty)$ and $G_i^\vartheta = \mathbf{1}_{t \geq \vartheta}$, $i = 1, 2$, are an equilibrium in the subgame at ϑ .

The resulting payoffs are $V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) = \max(F_\vartheta, M_\vartheta)$.

Remark 7.2. Where $L > F \geq M$, the choice of α_i^ϑ makes the respective other player indifferent between stopping and waiting, and $\alpha_i^\vartheta(\cdot)$ is right-continuous, allowing a limit outcome argument. Where $M > F$, stopping is of course the unique best reply. In the polar case $L_t = F_t = M_t$, there might not be a right-hand limit of $\mathbf{1}_{L_t > F_t} \frac{L_t - F_t}{L_t - M_t}$, so we set $\alpha_i^\vartheta(t) = 1$. If the limit does exist, one can use it to make $\alpha_i^\vartheta(\cdot)$ right-continuous even here, since the players will be completely indifferent in this case.

If $L_\vartheta = F_\vartheta > M_\vartheta$, each player becomes leader or follower with probability $\frac{1}{2}$.³⁵ This is the same outcome as in Fudenberg and Tirole (1985) for their smooth, deterministic model. If $L_\vartheta > F_\vartheta > M_\vartheta$, however, there is a positive probability of simultaneous stopping, which is the price of preemption, driving the payoffs down to F_ϑ .

7.2 General symmetric equilibria

We can now combine the equilibria we obtained for $F \geq L$ and $L > F$, respectively. With standard mixed strategies, the equilibria for a current second-mover advantage of Theorem 5.1 depend on the “endpoint condition” $\Delta G_i^\vartheta(M - F) \geq 0$, e.g., where a preemption regime begins with both players trying to stop immediately. Proposition 7.1, however, gives us “continuation” equilibria of immediate stopping at such transitions with payoffs $\max(F, M)$. Indeed, if player j uses an *extended* mixed strategy, the payoff difference for player i between stopping and waiting where G_j^ϑ jumps to 1 at $\hat{\tau}_j^\vartheta = \inf\{t \geq \vartheta \mid \alpha_j^\vartheta(t) > 0\}$ changes from $\Delta G_j^\vartheta(M - F)$ to

$$\Delta G_j^\vartheta(\alpha_j^\vartheta M + (1 - \alpha_j^\vartheta)L - F),$$

which is nonnegative if α_j^ϑ is as in Proposition 7.1. Therefore, this possibility to coordinate partially in preemption also generates suitable endpoints for attrition regimes where we cannot have $M \geq F$ before reaching $\{L > F\}$.

³⁴The extension to $\{M > F\}$ is straightforward in the symmetric case.

³⁵Then the \liminf and \limsup in Definition C.1 are both $\frac{1}{2}$ with the strategies of Proposition 7.1.

Now, for any symmetric stopping game – where the payoff processes L , F and M do not depend on the individual players – there exists a payoff-symmetric subgame-perfect equilibrium:

Theorem 7.3. *Under Assumption 2.1 there exists a payoff-symmetric subgame-perfect equilibrium in extended mixed strategies (G_1, α_1) , (G_2, α_2) given as follows:*

Pick $i, j \in \{1, 2\}$, $i \neq j$. For any $\vartheta \in \mathcal{T}$, set $\tau^\vartheta := \inf\{t \geq \vartheta \mid L_t > F_t \text{ or } M_t > F_t\}$. Define G_i^ϑ , G_j^ϑ as in Theorem 5.1 and $\alpha_i^\vartheta = \alpha_j^\vartheta$ as in Proposition 7.1.

Further, if the payoff processes are such that for any stopping time $\tau \in \mathcal{T}$ with $L_\tau = F_\tau$, either $F_\tau = M_\tau$ or $\tau = \inf\{t > \tau \mid L_t > F_t\}$ a.s., then there is a symmetric subgame-perfect equilibrium using G_i^ϑ from Theorem 5.1 for both $i = 1, 2$ and any $\vartheta \in \mathcal{T}$.

Proof: In Appendix B.1.

The idea of these equilibria is virtually pasting the war of attrition that we have on $\{F \geq L\}$ using the continuous strategies by Theorem 5.1 with the preemption equilibria of immediate stopping on $\{L > F\}$ by extended mixed strategies as in Proposition 7.1.

By the upper-semi-continuity of Assumption 2.1 (iii), it is during attrition feasible for the players to coordinate on a future continuation equilibrium with payoffs $\max(F, M)$. Then there will be no predictable drop in payoffs from setting off preemption. The corresponding equilibrium payoffs are

$$V_1^\vartheta = V_2^\vartheta = \operatorname{ess\,sup}_{\vartheta \leq \tau \in \mathcal{T}} E \left[\mathbf{1}_{\tau < \tau^\vartheta} L_\tau + \mathbf{1}_{\tau \geq \tau^\vartheta} \max(F_{\tau^\vartheta}, M_{\tau^\vartheta}) \mid \mathcal{F}_\vartheta \right]$$

with $\tau^\vartheta = \inf\{t \geq \vartheta \mid L_t > F_t \text{ or } M_t > F_t\}$ for any $\vartheta \in \mathcal{T}$.

While the endpoint condition $\Delta G_i^\vartheta(M - F) \geq 0$ at τ^ϑ is now replaced by the preemption continuation equilibria, we still need to ensure the second one, $\Delta G_i^\vartheta(M - F) \leq 0$ at $\tau_i^{G, \vartheta}(1)$; the cap $\tau^\vartheta \wedge \inf\{t \geq \vartheta \mid M_t > F_t\}$ works generally, but there may also be alternative choices in more specific cases. The proof of Theorem 7.3 relies of course on those of Theorem 5.1 and Proposition 7.1. The main issue is that the former was formulated in a reduced setting with “standard” mixed strategies, so we establish a formal relation to the present setting with extended mixed strategies.

8 Efficient symmetric equilibrium

The equilibria of Theorem 7.3 involve maximal preemption – wherever $L > F$ – and thus have a relatively simple structure: the game ends as soon as there is a strict first-mover advantage. Preemption need not be that severe if there are future continuation equilibria with sufficiently high (expected) payoffs. In this section we identify equilibria with least

possible preemption, entailing the highest attainable equilibrium payoffs. We focus on the class of *payoff-symmetric equilibria*, which are the subgame-perfect equilibria with $V_1^\vartheta = V_2^\vartheta$ a.s. at any stopping time $\vartheta \in \mathcal{T}$. These have strong implications for equilibrium strategies. Then, in competitive games, where M is throughout the lowest payoff, equilibrium payoffs are at most what can be obtained from optimally stopping $\min(L, F)$ – no matter how players mix, possibly using public correlation (Proposition 8.1). This bound on equilibrium payoffs enables us then to identify inevitable preemption points: those where the leader payoff L exceeds any continuation equilibrium payoff. Theorem 8.2 formulates a corresponding algorithm and establishes the existence of “efficient” subgame-perfect equilibria.

It is quite clear that any stopping on $\{F > L\}$ must induce the lower payoff L if M is generally the worst. The basis of our argument is the more subtle result that players also cannot exploit $L > F$ by mixing in any payoff-symmetric equilibrium, even if they have no time constraint.

Proposition 8.1. *Suppose $M \leq \min(L, F)$. Then, in any payoff-symmetric equilibrium and for any $\vartheta \in \mathcal{T}$, $i, j \in \{1, 2\}$, $i \neq j$,*

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \leq U_{L \wedge F}(\vartheta) = \operatorname{ess\,sup}_{\vartheta \leq \tau \in \mathcal{T}} E[L_\tau \wedge F_\tau \mid \mathcal{F}_\vartheta],$$

where we can in fact restrict ourselves to stopping times $\tau \leq \inf\{t \geq \vartheta \mid G_i^\vartheta \vee G_j^\vartheta \geq 1\}$.

The proof of Proposition 8.1 in Appendix B.2 is based on the following important facts for any payoff-symmetric equilibrium (which do not depend on the assumption $M \leq \min(L, F)$, yet): First, the conditional stopping probabilities of the players must be the same on $\{F \neq L\}$ (Lemma B.2), since a player who stops with a higher conditional probability also becomes leader with a higher conditional probability, whereas the other becomes follower on that event. As one consequence, G_1^ϑ and G_2^ϑ must then even be identical before they put any mass on $\{F = L\}$ (Lemma B.3). As another consequence, on $\{F \neq L\}$ we can only have simultaneous jumps. These are only possible if $M \geq F$, or if preemption occurs with $L \geq F > M$. Most importantly, we cannot have any jumps where $F > \max(L, M)$. Finally, the local payoff from any terminal jump is bounded by $\max(F, M)$ (Lemma B.4).

The intuition for Proposition 8.1 is now the following. In any equilibrium, player i must be willing to wait until any point at which $G_i^\vartheta < 1$, and stop only from there on with the corresponding conditional probabilities. Consider as such a point the first time at which any player puts some mass on $\{F \geq L\}$; call it $\tilde{\tau}$. By waiting until $\tilde{\tau}$, player i might become follower where G_j^ϑ increases earlier on $\{F < L\}$. At $\tilde{\tau}$, by definition at least one player is willing to stop. The corresponding (symmetric) local payoff is clearly $L_{\tilde{\tau}} \leq F_{\tilde{\tau}}$ if $G_1^\vartheta, G_2^\vartheta$ are continuous; a jump can only occur if indeed $F_{\tilde{\tau}} = L_{\tilde{\tau}}$, which is also the maximal local payoff (with the hypothesis $M \leq \min(L, F)$, we cannot have any jump where $F_{\tilde{\tau}} > L_{\tilde{\tau}}$ as we have

seen). Finally, it might happen that G_i^ϑ is exhausted on $\{F < L\}$, before ever reaching $\tilde{\tau}$. Then, however, we must have $G_1^\vartheta = G_2^\vartheta$. If they jump to 1, the terminal payoff is at most $F < L$; if they attain 1 continuously, this means in the limit becoming follower for sure on $\{F < L\}$. In summary, player i never receives more than $\min(L, F)$ where stopping occurs.

Proposition 8.1 implies that whenever $L_\vartheta > U_{L \wedge F}(\vartheta)$, we must have $G_1^\vartheta(\vartheta) \vee G_2^\vartheta(\vartheta) = 1$ by preemption.³⁶ If there are any such preemption points in the future, they also restrict the feasible stopping times τ to maximize the expected value of $\min(L, F)$ in Proposition 8.1, which even further reduces the maximally attainable equilibrium payoff. By iteration we can identify where preemption is inevitable.

Theorem 8.2. *Suppose $M \leq \min(L, F)$. Then there exists a maximal payoff-symmetric equilibrium with value*

$$V_1^\vartheta = V_2^\vartheta = \operatorname{ess\,sup}_{\tau \in [\vartheta, \tilde{\tau}(\vartheta)]} E[L_\tau \wedge F_\tau \mid \mathcal{F}_\vartheta] := U_{(L \wedge F)^{\tilde{\tau}(\vartheta)}}(\vartheta)$$

for any $\vartheta \in \mathcal{T}$, where $\tilde{\tau}(\vartheta)$ is the latest sustainable preemption point after ϑ determined by the following algorithm:

- (i) Set $\tau_0(\vartheta) := \inf\{t \geq \vartheta \mid L > U_{L \wedge F}\}$ and

$$(L \wedge F)^{\tau_0(\vartheta)} := (L_{t \wedge \tau_0(\vartheta)} \wedge F_{t \wedge \tau_0(\vartheta)})_{t \geq 0}$$

with Snell envelope $U_{(L \wedge F)^{\tau_0(\vartheta)}} = (\operatorname{ess\,sup}_{t \leq \tau \in \mathcal{T}} E[(L \wedge F)_\tau^{\tau_0(\vartheta)} \mid \mathcal{F}_t])_{t \geq 0}$.

- (ii) For all $n \in \mathbb{N}$ set $\tau_n(\vartheta) := \inf\{t \geq \vartheta \mid L > U_{(L \wedge F)^{\tau_{n-1}(\vartheta)}}\} \wedge \tau_{n-1}(\vartheta)$ and

$$(L \wedge F)^{\tau_n(\vartheta)} := (L_{t \wedge \tau_n(\vartheta)} \wedge F_{t \wedge \tau_n(\vartheta)})_{t \geq 0}$$

with Snell envelope $U_{(L \wedge F)^{\tau_n(\vartheta)}} = (\operatorname{ess\,sup}_{t \leq \tau \in \mathcal{T}} E[(L \wedge F)_\tau^{\tau_n(\vartheta)} \mid \mathcal{F}_t])_{t \geq 0}$.

- (iii) Take the monotone limit $\tilde{\tau}(\vartheta) := \lim_{n \rightarrow \infty} \tau_n(\vartheta)$.

Proof: In Appendix B.3.

The payoff-maximal equilibrium is implemented using the strategies of Theorem 7.3, but setting $\alpha_i^\vartheta = 0$ before $\tilde{\tau}(\vartheta)$. Constructing $\tilde{\tau}(\vartheta)$ by the algorithm is technically not difficult. The main problem is rather to verify the claimed equilibrium properties: to make sure that there is no preemption incentive where $L > F$ on $[\vartheta, \tilde{\tau}(\vartheta))$, that $\tilde{\tau}(\vartheta)$ is indeed maximal, and that there is a continuation equilibrium of preemption at $\tilde{\tau}(\vartheta)$. Further, measurability is a

³⁶This argument is not impaired by any jump $\Delta G_j^\vartheta(\vartheta) \in (0, 1)$ due to which player i could not realize L_ϑ . L is right-continuous, so player i could try to stop right after ϑ . The formal argument is given in the proof of Theorem 8.2.

major technical issue, since we want to have a time-consistent version of the strategies where we set $\alpha_i^\vartheta = 0$ on $[\vartheta, \tilde{\tau}(\vartheta))$ for all $\vartheta \in \mathcal{T}$, to achieve the maximal payoff in all subgames.

In order to suppress preemption where $L_\vartheta > F_\vartheta$, it is obviously not sufficient that there exists $\tau \geq \vartheta$ such that $E[L_\tau \wedge F_\tau | \mathcal{F}_\vartheta] \geq L_\vartheta$; this relation then rather has to hold on all of $[\vartheta, \tau] \cap \{L > F\}$. For instance, the algorithm of Theorem 8.2 can be applied to the model of Fudenberg and Tirole (1985) by visual inspection, shown in Figure 1.

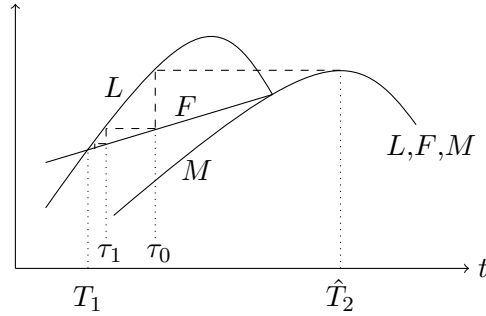


Figure 1: Preemption, Fudenberg and Tirole (1985)

L exceeds the future maximum of $\min(L, F)$ for the first time at τ_0 , whence there will be preemption. Taking that into account, at most the maximum of $\min(L, F)$ up to τ_0 might be achieved. However, L will also exceed this reduced value, at τ_1 . In the limit, $\tilde{\tau}(0) = T_1$ is the first inevitable preemption point. Fudenberg and Tirole also consider another, Case B, in which the peak at \hat{T}_2 is higher than first one. Then $\tilde{\tau}(0) = \infty$, because $L = F$ at and from their global peak onwards, and the player can coordinate on joint late adoption. In general we may have much more complex stochastic patterns, of course, with arbitrary regions of first- and second-mover advantages, that may trigger preemption or not.

9 Conclusion

In many timing games mixed strategies play an important role as we have argued, either for equilibrium existence or to resolve any strategic conflicts (about roles with differing amenities) within the game. Having analysed the two different kinds of local strategic incentives, we have been able to prove existence of and to characterize quite explicitly subgame-perfect equilibria for general symmetric stochastic timing games, providing symmetric equilibrium payoffs. Our approach is based on the general theory of optimal stopping and demonstrates which kinds of stopping problems need to be solved to verify equilibria, not only but in particular for mixed strategies.

There are possibly different equilibria for a given timing game, with varying degrees of preemption. We have considered the two extreme cases: If one initiates preemption whenever there is any first-mover advantage, payoffs may be severely restricted. However, we have shown

how to reduce preemption to a minimum and proved existence of corresponding equilibria with maximally attainable payoffs. If preemption can indeed be prevented in a certain regime with first-mover advantage (by sufficiently profitable future continuation equilibria), then there may also exist further equilibria with continuous mixing, which we have only employed at second-mover advantages. Nevertheless, any such additional mixing will be inefficient and induce lower payoffs (which one can also show directly).

A more specific strategic investment model with random first- and second-mover advantages is analysed in Steg and Thijssen (2015), where the strategies corresponding to the ones derived here have Markovian representations.

A Technical results

Lemma A.1. *A measurable process $X = (X_t)_{t \in \mathbb{R}_+}$ is of class (D) iff the set $\{X_\tau : \tau \text{ a stopping time}\}$ is uniformly integrable for any given $X_\infty \in L^1(P)$.*

Proof. We only need to show necessity: Let X be of class (D) and fix arbitrary $X_\infty \in L^1(P)$ and let \mathcal{T} denote the set of all stopping times. Then, for any $\tau \in \mathcal{T}$ and $n \in \mathbb{N}$, $|X_{\tau \wedge n}| \mathbf{1}_{\tau < \infty} \leq |X_{\tau \wedge n}|$. Hence the set $\{|X_{\tau \wedge n}| \mathbf{1}_{\tau < \infty} : \tau \in \mathcal{T}, n \in \mathbb{N}\} \cup \{|X_\tau| : \tau < \infty \in \mathcal{T}\}$ is uniformly integrable as well. As we may also include limits in probability of its elements, and $|X_\tau| \mathbf{1}_{\tau < \infty} = \lim_{n \rightarrow \infty} |X_{\tau \wedge n}| \mathbf{1}_{\tau < \infty}$ a.s. for any $\tau \in \mathcal{T}$, we observe that $\{|X_\tau| \mathbf{1}_{\tau < \infty} : \tau \in \mathcal{T}\}$ is uniformly integrable.

With $X_\infty \in L^1(P)$, also $\{|X_\infty| \mathbf{1}_{\tau = \infty} : \tau \in \mathcal{T}\}$ is uniformly integrable. Now let $\frac{\varepsilon}{2} > 0$. By uniform integrability there exists a $\delta > 0$ such that $\max\{E[|X_\tau| \mathbf{1}_{\tau < \infty} \mathbf{1}_A], E[|X_\infty| \mathbf{1}_{\tau = \infty} \mathbf{1}_A]\} \leq \frac{\varepsilon}{2}$ for any measurable A with $P(A) < \delta$ and any $\tau \in \mathcal{T}$. Hence we have $E[(|X_\tau| \mathbf{1}_{\tau < \infty} + |X_\infty| \mathbf{1}_{\tau = \infty}) \mathbf{1}_A] \leq \varepsilon$, which shows that $\{|X_\tau| \mathbf{1}_{\tau < \infty} + |X_\infty| \mathbf{1}_{\tau = \infty} : \tau \in \mathcal{T}\}$ is uniformly integrable as claimed. \square

Lemma A.2. *If L is a (measurable) process of class (D) then there exists a constant $K \in \mathbb{R}_+$ such that for any process G that is a.s. right-continuous, non-decreasing, non-negative and bounded by some $G_\infty \in L^\infty(P)$ and all random variables $0 \leq a \leq b \leq \infty$ a.s. we have*

$$(i) \quad E \left[\int_{[a,b)} |L_t| dG_t \right] \leq K \|G_\infty\|_\infty < \infty \quad \text{and}$$

$$(ii) \quad \int_{[a,b)} |L_t| dG_t = \int_0^\infty |L_{\tau^G(x)}| \mathbf{1}_{\tau^G(x) \in [a,b)} dx < \infty \quad \text{a.s.},$$

where $\tau^G(x) := \inf\{t \geq 0 \mid G_t \geq x\}$, $x \in \mathbb{R}_+$, and $\Delta G_0 \equiv G_0$; equivalently, “ $G_t > x$ ” in $\tau^G(x)$.

If $\{|L_\tau| \mathbf{1}_{\tau < \infty} : \tau \in \mathcal{T}\}$ is bounded in $L^\infty(P)$ by $K \in \mathbb{R}_+$ and G bounded by some $G_\infty \in L^1(P)$, (i) holds with $KE[G_\infty]$ instead and (ii) as stated.

Proof. The (a.s.) non-decreasing family of stopping times $(\tau^G(x))_{x \in \mathbb{R}_+}$ is the left-continuous inverse of G , which satisfies

$$\tau^G(x) \leq t \Leftrightarrow G_t \geq x.$$

Thus, with the convention $\int_{[0,c]} dG = G_c$, we have $\int_{[0,\infty)} \mathbf{1}_A dG = \int_0^\infty \mathbf{1}_{\tau^G(x) \in A} dx$ for all $A \in \{[0,c] : c \in \mathbb{R}_+\}$ and hence for $A = \mathbb{R}_+$ by monotone convergence, a.s. By a monotone class argument³⁷ the relation holds on all of $\mathcal{B}(\mathbb{R}_+)$ a.s.

Since $L(\omega) : \mathbb{R}_+ \rightarrow \mathbb{R}$, $t \mapsto L_t(\omega)$, is Borel measurable³⁸ like the function $\mathbf{1}_{t \in [a(\omega), b(\omega))}$, we now obtain the following change-of-variable formula³⁹:

$$\int_{[a,b)} |L_t| dG_t = \int_{\{\tau^G(x) < \infty\}} |L_{\tau^G(x)}| \mathbf{1}_{\tau^G(x) \in [a,b)} dx \quad \text{a.s.}$$

As $\inf\{t \geq 0 \mid G_t > x\} = \tau^G(x+)$, which differs from $\tau^G(x)$ only on a set of Lebesgue measure (dx) 0, we can equivalently use the former. By Fubini's Theorem

$$E \left[\int_0^\infty |L_{\tau^G(x)}| \mathbf{1}_{\tau^G(x) \in [a,b)} dx \right] \leq \int_0^{\|G_\infty\|_\infty} E \left[|L_{\tau^G(x)}| \mathbf{1}_{\tau^G(x) < \infty} \right] dx. \quad (\text{A.1})$$

As L is of class (D), $\{|L_\tau| \mathbf{1}_{\tau < \infty} : \tau \in \mathcal{T}\}$ is bounded in $L^1(P)$ by some $K < \infty$, whence the RHS of (A.1) is bounded by $K\|G_\infty\|_\infty$ if the latter is finite. If $\sup_{\tau \in \mathcal{T}} \|L_\tau \mathbf{1}_{\tau < \infty}\|_\infty \leq K$ and G bounded by $G_\infty \in L^1(P)$, then the RHS of (A.1) is bounded by

$$\int_0^\infty E \left[K \mathbf{1}_{\tau^G(x) < \infty} \right] dx = E \left[\int_0^\infty K \mathbf{1}_{\tau^G(x) < \infty} dx \right] \leq KE[G_\infty] < \infty.$$

In either case it follows that $\int_{[a,b)} |L_t| dG_t < \infty$ a.s. □

Lemma A.3. *Suppose the processes L , F , and M are optional and of class (D) and the adapted process G is right-continuous, non-decreasing and taking values in $[0, 1]$ a.s. Then S defined by*

$$S_u := \int_{[0,u)} F_s dG_s + \Delta G_u M_u + (1 - G_u) L_u, \quad u \in \mathbb{R}_+,$$

is optional and of class (D) as well.

Proof. The components of S are obviously optional, in particular the integral being a left-continuous and ΔG the difference of a right-continuous and a left-continuous adapted process. We have to show that $(S(\tau))_{\tau < \infty}$ a.s. is uniformly integrable and obviously only need to consider the first, integral component of S . By the Theorem of de la Vallée-Poussin it is necessary

³⁷See, e.g., Kallenberg (2002), Theorem 1.1.

³⁸See, e.g., Kallenberg (2002), Lemma 1.26 (i).

³⁹See, e.g., Kallenberg (2002), Lemma 1.22. One needs to restrict dx to $\{\tau^G(x) < \infty\}$, which is redundant whenever we have $[a, b)$.

and sufficient for a process X to be of class (D) that there exists a non-decreasing and convex function $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lim_{t \rightarrow \infty} \frac{g(t)}{t} = \infty$ and $\sup_{\tau < \infty} E[g(|X_\tau|)] < \infty$.

Borrowing the corresponding function g that there is for F and by a change of variable as in Lemma A.2 we obtain

$$\begin{aligned} \sup_{\tau < \infty} E \left[g \left(\left| \int_{[0, \tau)} F_s dG_s \right| \right) \right] &\leq \sup_{\tau < \infty} E \left[g \left(\int_0^1 |F_{\tau^G(x)}| \mathbf{1}_{\tau^G(x) < \tau} dx \right) \right] \\ &\leq \sup_{\tau < \infty} E \left[\int_0^1 g(|F_{\tau^G(x)}|) dx \right] = \int_0^1 \underbrace{E[g(|F_{\tau^G(x)}|)]}_{\leq \sup_{\sigma < \infty} E[g(|F_\sigma|)] < \infty} dx < \infty. \end{aligned}$$

We used Jensen's inequality for the last estimate. \square

Lemma A.4. *Consider two stopping times $\sigma \leq \tau$ and an event $C \in \mathcal{F}_\sigma$. Then*

$$\vartheta := \sigma \mathbf{1}_C + \tau \mathbf{1}_{C^c}$$

is a stopping time. If the filtration is complete, it suffices that $\sigma \leq \tau$ a.s.

Proof. To verify that ϑ is a stopping time, we check whether $\{\vartheta \leq u\} \in \mathcal{F}_u$ for all $u \in \mathbb{R}_+$. First note that

$$\{\vartheta \leq u\} = (\{\sigma \leq u\} \cap C) \cup (\{\tau \leq u\} \cap C^c).$$

The first intersection belongs to \mathcal{F}_u by definition of \mathcal{F}_σ . Then also its complement $(\{\sigma \leq u\} \cap C)^c = \{\sigma > u\} \cup C^c \in \mathcal{F}_u$ and thus $C^c \cap \{\sigma \leq u\} \in \mathcal{F}_u$. Finally, as $\{\tau \leq u\} \in \mathcal{F}_u$, we conclude

$$C^c \cap \{\sigma \leq u\} \cap \{\tau \leq u\} = C^c \cap \{\tau \leq u\} \in \mathcal{F}_u.$$

If $\sigma \leq \tau$ only a.s., the last equality holds up to the nullset $C^c \cap \{\sigma > u\} \cap \{\tau \leq u\}$, which is contained in \mathcal{F}_u if the filtration is complete. \square

Lemma A.5. *The process $\tilde{L}^{\tau^\vartheta} := \mathbf{1}_{t < \tau^\vartheta} L + \mathbf{1}_{t \geq \tau^\vartheta} \max(F_{\tau^\vartheta}, M_{\tau^\vartheta})$ defined in Theorem 5.1 is upper-semi-continuous from the left in expectation on $[\vartheta, \infty]$, where $D_{\tilde{L}}$ is hence left-continuous a.s.*

Proof. Let (τ_n) be a sequence of stopping times that is a.s. increasing and taking values in $[\vartheta, \infty]$, and denote the limit by $\tau \in \mathcal{T}$. Define the measurable set

$$A := \bigcap_n \{\tau_n < \tau^\vartheta\}.$$

Then $\lim_n \tilde{L}_{\tau_n} = \tilde{L}_\tau$ a.s. on A^c , implying $\lim_n E[\mathbf{1}_{A^c} \tilde{L}_{\tau_n}] = E[\mathbf{1}_{A^c} \tilde{L}_\tau]$ since \tilde{L} is of class (D).

We obtain the analogue for $L \wedge F$ if we use $(\tau_n \wedge \tau^\vartheta)$ and $(\tau \wedge \tau^\vartheta)$, respectively. Combining the latter fact with upper-semi-continuity from the left in expectation given by Assumption

2.1 (iii) yields

$$\begin{aligned} \limsup_n E[\mathbf{1}_A(L \wedge F)_{\tau_n \wedge \tau^\vartheta} + \mathbf{1}_{A^c}(L \wedge F)_{\tau_n \wedge \tau^\vartheta}] &\leq E[\mathbf{1}_A(L \wedge F)_{\tau \wedge \tau^\vartheta} + \mathbf{1}_{A^c}(L \wedge F)_{\tau \wedge \tau^\vartheta}] \\ \Rightarrow \limsup_n E[\mathbf{1}_A(L \wedge F)_{\tau_n}] &\leq E[\mathbf{1}_A(L \wedge F)_\tau]. \end{aligned}$$

Note that $(L \wedge F)_{\tau_n} = \tilde{L}_{\tau_n}$ on A and $(L \wedge F)_\tau = \tilde{L}_\tau$ on $A \subset \{\tau \leq \tau^\vartheta\}$, implying the first claim.

Finally we show that if \tilde{L} is upper-semi-continuous from the left in expectation on an interval $[\vartheta, \infty]$, then the paths of $D_{\tilde{L}}$ are left-continuous on that interval, a.s. Define as an auxiliary process $\hat{L} := \mathbf{1}_{t < \vartheta} \hat{M} + \mathbf{1}_{t \geq \vartheta} \tilde{L}$, where \hat{M} is a right-continuous version of the martingale $(E[\tilde{L}_\vartheta | \mathcal{F}_t])_{t \geq 0}$, which is uniformly integrable and thus (left-) continuous in expectation thanks to optional sampling. \hat{L} inherits upper-semi-continuity from the left in expectation because $E[\hat{L}_{\tau_n}] = E[\mathbf{1}_{\tau_n < \vartheta} \tilde{L}_\vartheta + \mathbf{1}_{\tau_n \geq \vartheta} \tilde{L}_{\tau_n}] = E[\tilde{L}_{\tau_n \vee \vartheta}]$ and similarly $E[\hat{L}_\tau] = E[\tilde{L}_{\tau \vee \vartheta}]$. Since \tilde{L} and \hat{L} agree on $[\vartheta, \infty]$, their Snell envelopes $U_{\tilde{L}}$ and $U_{\hat{L}}$ agree at any stopping time in that interval and thus $U_{\tilde{L}} \mathbf{1}_{t \geq \vartheta}$ and $U_{\hat{L}} \mathbf{1}_{t \geq \vartheta}$ are indistinguishable by the uniqueness of optional projections. The same holds for the compensators $D_{\tilde{L}}$ and $D_{\hat{L}}$ on $[\vartheta, \infty]$ by uniqueness of the Doob-Meyer-decomposition. $D_{\hat{L}}$ is left-continuous a.s. since \hat{L} is upper-semi-continuous from the left in expectation, see footnote 24. \square

Lemma A.6. *Let $A, B: \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{+\infty\}$ be two right-continuous, nondecreasing functions, $0 \leq \Delta B \leq 1$. Then the differential equation*

$$dG = (1 - G) dA, \quad G_{0-} = a \in \mathbb{R} \tag{A.2}$$

has the solution

$$G = 1 - (1 - a)e^{-\int dA^c} \prod (1 + \Delta A)^{-1} = 1 - (1 - a)e^{-\int dA^c - \sum \ln(1 + \Delta A)},$$

where $A^c = A - \sum \Delta A \in [A_0, A]$ is the continuous part of A , and the differential equation

$$dG = (1 - G_-) dB, \quad G_{0-} = b \in \mathbb{R} \tag{A.3}$$

has the solution

$$G = 1 - (1 - b)e^{-\int dB^c} \prod (1 - \Delta B) = 1 - (1 - b)e^{-\int dB^c + \sum \ln(1 - \Delta B)}.$$

Any solution to (A.2) or (A.3) is monotone towards, but never crossing, 1. G solves both equations iff $\Delta A = \frac{\Delta B}{1 - \Delta B}$, resp. $\Delta B = \frac{\Delta A}{1 + \Delta A}$.

Proof. Straightforward to check. Note that $dG_0 = \frac{(1-a)dA_0}{1+\Delta A_0}$ for any solution to (A.2), implying

both monotonicity on either side of 1 and that the latter can never be crossed. Indeed, $G_0 \geq 1 \Leftrightarrow \Delta G_0 \geq 1 - G_{0-} = 1 - a \Leftrightarrow 0 \geq 1 - a$.

Similarly $dG_0 = (1-b)dB_0$ for any solution to (A.3), implying $G_0 \geq 1 \Leftrightarrow \Delta G_0 \geq 1 - G_{0-} = 1 - b \Leftrightarrow 0 \geq 1 - b$ and $\Delta B_0 < 1$, where $\Delta B_0 \in [0, 1]$ is used for the last equivalence. \square

Lemma A.7. *Let $(Y_t, Z_t)_{t \in [0,1]}$ be a family of random variables. Assume that the family (Y_t) is uniformly integrable and that (Z_t) is bounded in $L^\infty(P)$ and $Z_t \rightarrow 0$ in probability as $t \rightarrow 1$. Then*

$$\lim_{t \rightarrow 1} E[Y_t Z_t] = 0.$$

Proof. $(\|Z_t\|_\infty)$ is bounded by a constant K , hence (Z_t) is uniformly integrable and converges to the constant 0 also in $L^1(P)$ as $t \rightarrow 1$. Since (Y_t) is uniformly integrable, we can find for any $\varepsilon > 0$ a suitable constant $K_\varepsilon \geq 0$ such that

$$E[\mathbf{1}_{|Y_t| \geq K_\varepsilon} |Y_t Z_t|] \leq \varepsilon K \quad \text{for all } t \in [0, 1].$$

In combination,

$$\begin{aligned} \limsup_{t \rightarrow 1} E[|Y_t Z_t|] &= \limsup_{t \rightarrow 1} \left\{ E[\mathbf{1}_{|Y_t| \geq K_\varepsilon} |Y_t Z_t|] + E[\mathbf{1}_{|Y_t| < K_\varepsilon} |Y_t Z_t|] \right\} \\ &\leq \limsup_{t \rightarrow 1} E[\mathbf{1}_{|Y_t| \geq K_\varepsilon} |Y_t Z_t|] + \lim_{t \rightarrow 1} E[\mathbf{1}_{|Y_t| < K_\varepsilon} |Y_t Z_t|] \leq \varepsilon K \end{aligned}$$

and the claim follows. \square

B Proofs

B.1 Proofs for results in Sections 3–7

Proof of Lemma 3.1. For G_i^ϑ feasible we can define the right-continuous inverse

$$\tau_i^{G, \vartheta}(x) := \inf\{s \geq \vartheta \mid G_i^\vartheta(s) > x\}, \quad x \in \mathbb{R}_+.$$

As in Lemma A.2 it leads to the change-of-variable formula

$$\int_{[\vartheta, \infty)} S_i^\vartheta(s) dG_i^\vartheta(s) = \int_0^1 S_i^\vartheta(\tau_i^{G, \vartheta}(x)) \mathbf{1}_{\tau_i^{G, \vartheta}(x) < \infty} dx \quad \text{a.s.}$$

Further we have $x > G_i^\vartheta(\infty-) \Rightarrow \tau_i^{G, \vartheta}(x) = \infty \Rightarrow x \geq G_i^\vartheta(\infty-)$, i.e., $\mathbf{1}_{x > G_i^\vartheta(\infty-)} \leq \mathbf{1}_{\tau_i^{G, \vartheta}(x) = \infty} \leq \mathbf{1}_{x \geq G_i^\vartheta(\infty-)}$ for all $x \in \mathbb{R}_+$ a.s., implying

$$\Delta G_i^\vartheta(\infty) S_i^\vartheta(\infty) = \left(\int_0^1 \mathbf{1}_{\tau_i^{G, \vartheta}(x) = \infty} dx \right) S_i^\vartheta(\infty) \quad \text{a.s.}$$

Thus,

$$\begin{aligned}
V_i^\vartheta(G_i^\vartheta, G_j^\vartheta) &= \int_0^1 E[S_i^\vartheta(\tau_i^{G,\vartheta}(x)) \mathbf{1}_{\tau_i^{G,\vartheta}(x) < \infty} \mid \mathcal{F}_\vartheta] dx + \int_0^1 E[S_i^\vartheta(\infty) \mathbf{1}_{\tau_i^{G,\vartheta}(x) = \infty} \mid \mathcal{F}_\vartheta] dx \\
&= \int_0^1 E[S_i^\vartheta(\tau_i^{G,\vartheta}(x)) \mid \mathcal{F}_\vartheta] dx \\
&\leq \int_0^1 \operatorname{ess\,sup}_{\vartheta \leq \tau \in \mathcal{T}} E[S_i^\vartheta(\tau) \mid \mathcal{F}_\vartheta] dx = \operatorname{ess\,sup}_{\vartheta \leq \tau \in \mathcal{T}} E[S_i^\vartheta(\tau) \mid \mathcal{F}_\vartheta]. \quad \square
\end{aligned} \tag{B.1}$$

Proof of Lemma 4.1. By right-continuity of L , we have $S_i^\vartheta(\tau+) - S_i^\vartheta(\tau) = \Delta G_j^\vartheta(\tau)(F_\tau - M_\tau) \geq 0$ for any $\tau \in \mathcal{T}$. Now consider the set $\{\tau_i < \tau_L(\vartheta)\}$ on which we have, a.s.,

$$\begin{aligned}
&E[S_i^\vartheta(\tau_L(\vartheta)+) - S_i^\vartheta(\tau_i) \mid \mathcal{F}_{\tau_i}] \\
&= E\left[\int_{[\tau_i, \tau_L(\vartheta)]} F_s dG_j^\vartheta(s) + (1 - G_j^\vartheta(\tau_L(\vartheta)))L_{\tau_L(\vartheta)} - \Delta G_j^\vartheta(\tau_i)M_{\tau_i} - (1 - G_j^\vartheta(\tau_i))L_{\tau_i} \mid \mathcal{F}_{\tau_i}\right] \\
&\geq E\left[F_{\tau_L(\vartheta)}(G_j^\vartheta(\tau_L(\vartheta)) - G_j^\vartheta(\tau_i-)) + (1 - G_j^\vartheta(\tau_L(\vartheta)))L_{\tau_L(\vartheta)} - (1 - G_j^\vartheta(\tau_i-))L_{\tau_i} \mid \mathcal{F}_{\tau_i}\right] \\
&\geq E\left[L_{\tau_L(\vartheta)}(1 - G_j^\vartheta(\tau_i-)) - (1 - G_j^\vartheta(\tau_i-))L_{\tau_i} \mid \mathcal{F}_{\tau_i}\right] \geq 0.
\end{aligned}$$

The first inequality is obtained from a change of variable (demonstrated below) similar to that in the proof of Lemma 3.1, exploiting that F is a supermartingale, and $L \geq M$. The second inequality is due to $F \geq L$ and the last to the optimality of $\tau_L(\vartheta)$. Note that the last one will be strict if $P[\tau^i < \tau_L(\vartheta) \text{ and } G_j^\vartheta(\tau_i-) < 1] > 0$ by suboptimality of any $\vartheta \leq \tau_i < \tau_L(\vartheta)$. The second claimed estimate of the lemma follows from setting $\tau_i = \vartheta$ in the steps above. Further, the previous and following steps go through identically with $\tau_L(\vartheta)$ replaced by $\tau^L(\vartheta)$.

The change of variable proceeds as follows:

$$\begin{aligned}
&E\left[\int_{[\tau_i, \tau_L(\vartheta)]} F_s dG_j^\vartheta(s) \mid \mathcal{F}_{\tau_i}\right] \\
&= E\left[\int_0^1 F_{\tau_j^{G,\vartheta}(x)} \mathbf{1}_{\tau_j^{G,\vartheta}(x) \in [\tau_i, \tau_L(\vartheta)]} dx \mid \mathcal{F}_{\tau_i}\right] = \int_0^1 E[F_{\tau_j^{G,\vartheta}(x)} \mathbf{1}_{\tau_j^{G,\vartheta}(x) \in [\tau_i, \tau_L(\vartheta)]} \mid \mathcal{F}_{\tau_i}] dx \\
&\geq \int_0^1 E[E[F_{\tau_L(\vartheta)} \mid \mathcal{F}_{\tau_j^{G,\vartheta}(x)}] \mathbf{1}_{\tau_j^{G,\vartheta}(x) \in [\tau_i, \tau_L(\vartheta)]} \mid \mathcal{F}_{\tau_i}] dx = \int_0^1 E[F_{\tau_L(\vartheta)} \mathbf{1}_{\tau_j^{G,\vartheta}(x) \in [\tau_i, \tau_L(\vartheta)]} \mid \mathcal{F}_{\tau_i}] dx \\
&= E\left[F_{\tau_L(\vartheta)} \int_0^1 \mathbf{1}_{\tau_j^{G,\vartheta}(x) \in [\tau_i, \tau_L(\vartheta)]} dx \mid \mathcal{F}_{\tau_i}\right] = E\left[F_{\tau_L(\vartheta)}(G_j^\vartheta(\tau_L(\vartheta)) - G_j^\vartheta(\tau_i-)) \mid \mathcal{F}_{\tau_i}\right]. \quad \square
\end{aligned}$$

Proof of Proposition 4.2. Let $\vartheta \in \mathcal{T}$ and $C \in \mathcal{F}_{\tau_L(\vartheta)}$. Then τ_1^*, τ_2^* as hypothesized are stopping times thanks to Lemma A.4, as $\tau_L(\vartheta) \leq \tau_F(\vartheta)$ a.s.

To verify the optimality of τ_1^* , it suffices by Lemmata 3.1 and 4.1 to consider stopping S_i^ϑ from $\tau_L(\vartheta)$. On C , $S_1^\vartheta(t) = L_{t \wedge \tau_F(\vartheta)}$ for all $t \geq \tau_L(\vartheta)$, such that stopping immediately at $\tau_L(\vartheta)$ is optimal by its optimality for L . On C^c , $S_1^\vartheta(t) = F_{\tau_L(\vartheta)} \geq M_{\tau_L(\vartheta)}$ for all $t > \tau_L(\vartheta)$,

with equality on $\{\tau_F(\vartheta) = \tau_L(\vartheta)\}$ by hypothesis. Hence, $\tau_F(\vartheta)$ is optimal on C^c . The same argument applies to τ_2^* , swapping C and C^c .

We can use $\tau_F(\vartheta) := \inf\{t \geq \vartheta \mid F_t = M_t\}$, since it does not occur before $\tau_L(\vartheta)$, a.s. Indeed, as F is a supermartingale dominating L , it also dominates the Snell envelope U_L . Therefore, at $\tau_F(\vartheta)$, $F = M$ (by right-continuity and $F \geq M$), implying $U_L = L$ by $L \geq M$. Hence, $\tau_F(\vartheta) \geq \inf\{t \geq \vartheta \mid U_L(t) = L_t\} = \tau_L(\vartheta)$. \square

Proof of Theorem 5.1. $\tilde{L}^{\tau^\vartheta}$ is right-continuous a.s. and of class (D), so it has a Snell envelope $U_{\tilde{L}}^{\tau^\vartheta}$ with an integrable and predictable compensator $D_{\tilde{L}}^{\tau^\vartheta}$. We write for simplicity \tilde{L} and $D_{\tilde{L}}$. The latter is continuous on $[\vartheta, \infty]$ a.s. since \tilde{L} there is upper-semi-continuous from the left in expectation, see Lemma A.5 and footnotes 24, 25.

Now G_i^ϑ is a feasible mixed strategy, as it is clearly adapted and a.s. right-continuous and non-decreasing, taking values $G_i^\vartheta = 0$ on $[0, \vartheta)$ and $G_i^\vartheta(\infty) = 1$. The only possible jump occurs at $\tau_i^{G, \vartheta}(1) := \inf\{t \geq \vartheta \mid G_i^\vartheta(t) = 1\}$.

G_j^ϑ as defined in (5.2) is even continuous up to τ^ϑ : $\mathbf{1}_{F > L}/(F - L)$ can be understood as a Radon-Nikodym derivative, such that the integral defines a measure on \mathbb{R}_+ , which is absolutely continuous with respect to the (finite) measure $dD_{\tilde{L}}$ having no mass points.⁴⁰

To prove first that G_j^ϑ is a best reply to G_i^ϑ we will show in view of Lemma 3.1 and its proof that

$$E[S_j^\vartheta(\tau^\vartheta) \mid \mathcal{F}_\vartheta] \geq E[S_j^\vartheta(\tau) \mid \mathcal{F}_\vartheta]$$

for all stopping times $\tau \geq \vartheta$, with equality whenever $dG_j^\vartheta > 0$ (implying equality in (B.1)). In fact, we establish the stronger condition

$$E[S_j^\vartheta(\tau^\vartheta) - S_j^\vartheta(\tau) \mid \mathcal{F}_\tau] \geq 0 \quad (\text{B.2})$$

(with equality whenever $dG_j^\vartheta(\tau) > 0$), where it suffices, however, to consider stopping times $\tau \leq \tau^\vartheta$, since $\Delta S_j^\vartheta(\tau^\vartheta) = \Delta G_i^\vartheta(\tau^\vartheta)(F_{\tau^\vartheta} - M_{\tau^\vartheta}) \leq 0$ by hypothesis, and S_j^ϑ is constant on $(\tau_i^{G, \vartheta}(1), \infty]$.

To ease readability in the following demonstration of (B.2), we simply write G_i for G_i^ϑ , S_j for S_j^ϑ , a for τ and b for $\tau_i^{G, \vartheta}(1)$. By the other hypothesis $\Delta G_i(F - M) \geq 0$ at $b < \tau^\vartheta$, now $S_j(\tau^\vartheta) = \int_{[0, b)} F dG_i + \Delta G_i(b) \max(F_b, M_b)$. Further, our G_i satisfies

$$dG_i(s) = (1 - G_i(s)) \frac{dD_{\tilde{L}}(s)}{F_s - L_s}$$

for all $s \in [a, b)$ a.s., implying

$$\int_{[a, b)} (F_s - L_s) dG_i(s) = \int_{[a, b)} (1 - G_i(s)) dD_{\tilde{L}}(s), \quad (\text{B.3})$$

⁴⁰The new measure is also σ -finite as $\{F > L\} = \bigcup_{n \in \mathbb{N}} \{F - L \geq \frac{1}{n}\}$.

where $\int L dG_i$ is well defined by Lemma A.2 for L of class (D). We apply integration by parts to the RHS (adjusting for $[a, b]$ closed on the left, open on the right, and recalling that $D_{\tilde{L}}$ is continuous) to find

$$\begin{aligned} \int_{[a,b)} (1 - G_i(s)) dD_{\tilde{L}}(s) &= \int_{[a,b)} D_{\tilde{L}}(s) dG_i(s) \\ &\quad + (1 - G_i(b-))D_{\tilde{L}}(b) - (1 - G_i(a-))D_{\tilde{L}}(a). \end{aligned} \quad (\text{B.4})$$

With the help of $G_i(b) = 1$, (B.3) and (B.4) we see that

$$\begin{aligned} S_j(\tau^\vartheta) - S_j(a) &= \int_{[a,b)} F_s dG_i(s) + \Delta G_i(b) \max(F_b, M_b) - (1 - G_i(a))L_a - \Delta G_i(a)M_a \\ &= \int_{[a,b)} (L_s + D_{\tilde{L}}(s)) dG_i(s) + (1 - G_i(b-))(\max(F_b, M_b) + D_{\tilde{L}}(b)) \\ &\quad - (1 - G_i(a-))(L_a + D_{\tilde{L}}(a)) + \Delta G_i(a)(L_a - M_a). \end{aligned}$$

Now, as the martingale component $M_{\tilde{L}}$ of the Snell envelope is uniformly integrable, $\int M_{\tilde{L}} dG_i$ is well defined by Lemma A.2. By the change of variable proposed there we find that

$$\begin{aligned} E \left[\int_{[a,b)} M_{\tilde{L}}(s) dG_i(s) \middle| \mathcal{F}_a \right] &= \int_0^1 E \left[M_{\tilde{L}}(\tau_i^G(x)) \mathbf{1}_{\tau_i^G(x) \in [a,b)} \middle| \mathcal{F}_a \right] dx \\ &= \int_0^1 E \left[E \left[M_{\tilde{L}}(b) \middle| \mathcal{F}_{\tau_i^G(x)} \right] \mathbf{1}_{\tau_i^G(x) \in [a,b)} \middle| \mathcal{F}_a \right] dx = E \left[M_{\tilde{L}}(b) \int_0^1 \mathbf{1}_{\tau_i^G(x) \in [a,b)} dx \middle| \mathcal{F}_a \right] \\ &= E \left[M_{\tilde{L}}(b)(G_i(b-) - G_i(a-)) \middle| \mathcal{F}_a \right] = -E \left[M_{\tilde{L}}(b)(1 - G_i(b-)) \middle| \mathcal{F}_a \right] + M_{\tilde{L}}(a)(1 - G_i(a-)). \end{aligned}$$

Combining the last two results yields

$$\begin{aligned} E[S_j(\tau^\vartheta) - S_j(a) \middle| \mathcal{F}_a] &= E \left[\int_{[a,b)} (L_s + D_{\tilde{L}}(s) - M_{\tilde{L}}(s)) dG_i(s) + (1 - G_i(b-))(\max(F_b, M_b) + D_{\tilde{L}}(b) - M_{\tilde{L}}(b)) \right. \\ &\quad \left. - (1 - G_i(a-))(L_a + D_{\tilde{L}}(a) - M_{\tilde{L}}(a)) + \Delta G_i(a)(L_a - M_a) \middle| \mathcal{F}_a \right]. \end{aligned} \quad (\text{B.5})$$

It remains to observe that $L + D_{\tilde{L}} - M_{\tilde{L}} = \tilde{L} - U_{\tilde{L}} = 0$ whenever $dD_{\tilde{L}} > 0$, i.e., when $dG_i > 0$ on $[a, b)$, which makes the integral vanish; cf. (3.6). The same argument applies to the second term where $\Delta G_i(b) > 0$: if $b < \tau^\vartheta$, then the jump must result from $dD_{\tilde{L}}(b) > 0$ and $F_b = L_b = \tilde{L}_b = U_{\tilde{L}}(b)$ ($\geq M_b$ by hypothesis); if $b = \tau^\vartheta$, then $\max(F_b, M_b) = \tilde{L}_b = U_{\tilde{L}}(b)$ as \tilde{L} is constant on $[\tau^\vartheta, \infty]$. As $\Delta G_i(a) = 0$ on $\{a < b\}$, we are here left with

$$E[S_j(\tau^\vartheta) - S_j(a) \middle| \mathcal{F}_a] = (1 - G_i(a-))(U_{\tilde{L}}(a) - L_a) \geq 0, \quad (\text{B.6})$$

with equality whenever $dG_i(a) > 0$. On $\{a = b\}$, we collect terms to

$$E[S_j(\tau^\vartheta) - S_j(a) | \mathcal{F}_a] = \Delta G_i(b)(U_{\bar{L}}(b) - M_b) \geq 0 \quad (\text{B.7})$$

due to $U_{\bar{L}}(b) = \max(F_b, M_b) \geq M_b$ a.s., as we have argued before. On $\{b < \tau^\vartheta\}$, dG_j^ϑ puts no mass on $[b]$. On $\{b = \tau^\vartheta\}$, (B.7) is binding iff $\Delta G_i^\vartheta(M - F) \geq 0$ a.s. at $\tau^\vartheta < \infty$ (for necessity of this condition for equilibrium note that $\Delta G_i^\vartheta(\tau^\vartheta) > 0 \Rightarrow \Delta G_j^\vartheta(\tau^\vartheta) > 0$). This establishes (B.2).

In the case that $\Delta G_i^\vartheta(M - F) = 0$ a.s. at $\inf\{t \in \mathbb{R}_+ | G_i^\vartheta(t) = 1\} < \infty$, the identical arguments show that $G_j^\vartheta = G_i^\vartheta$ is a best reply to itself, because then S_j^ϑ is constant on $[\tau_i^{G, \vartheta}(1), \infty]$ (i.e., $S_j(\tau^\vartheta) = S_j(b)$ in (B.7)).

There are some slight variations to the above in proving that G_i^ϑ is a best reply to $G_j^\vartheta \neq G_i^\vartheta$ without the previous additional condition. The analogue to (B.2) that we seek is

$$E[S_i^\vartheta(\tau_i^{G, \vartheta}(1)) - S_i^\vartheta(\tau) | \mathcal{F}_\tau] \geq 0 \quad (\text{B.8})$$

for all stopping times $\tau \in [\vartheta, \tau_i^{G, \vartheta}(1))$, with equality whenever $dG_i^\vartheta > 0$. Afterwards we will show that at $\tau_i^{G, \vartheta}(1)$ it is optimal to stop immediately.

To derive (B.8) we can apply similar arguments as above. The main difference is that switching to $S_i = S_i^\vartheta$ and $G_j = G_j^\vartheta$ while keeping $b = \tau_i^{G, \vartheta}(1)$, we may have $G_j(b) < 1$. Nevertheless, $\Delta G_j(b)M_b = \Delta G_j(b) \max(F_b, M_b)$ (in particular $\Delta G_j(b) = 0$ on $\{b < \tau^\vartheta\} \cup \{\Delta G_i(b) = 0\}$), so that on the one hand $S_i(b) = S_j(\tau^\vartheta)$. Indeed, $G_i = G_j$ on $[0, b)$, so $S_i(b) - S_j(\tau^\vartheta) = (1 - G_j(b))L_b + (\Delta G_j(b) - \Delta G_i(b)) \max(F_b, M_b) = 0$ on $\{b < \tau^\vartheta\} \cap \{\Delta G_i(b) > 0\}$ – the only set where they might differ – but there $L_b = F_b$ ($\geq M_b$ by hypothesis) and $G_j(b-) = G_i(b-)$. This implies payoff symmetry once we have (B.8). On the other hand we get analogously to above (with possibly $G_j(b) < 1$)

$$\begin{aligned} E[S_i(b) - S_i(a) | \mathcal{F}_a] &= E \left[\int_{[a, b)} (L_s + D_{\bar{L}}(s) - M_{\bar{L}}(s)) dG_j(s) + \Delta G_j(b)(\max(F_b, M_b) - L_b) \right. \\ &\quad \left. + (1 - G_j(b-))(L_b + D_{\bar{L}}(b) - M_{\bar{L}}(b)) \right. \\ &\quad \left. - (1 - G_j(a-))(L_a + D_{\bar{L}}(a) - M_{\bar{L}}(a)) + \Delta G_j(a)(L_a - M_a) \right] | \mathcal{F}_a. \end{aligned} \quad (\text{B.9})$$

The integral vanishes as before. Since b is still the same, on $\{b < \tau^\vartheta\}$ again $F_b = L_b = U_{\bar{L}}(b) \geq M_b$; on $\{b = \tau^\vartheta\}$ again $\max(F_b, M_b) = U_{\bar{L}}(b)$ and $\Delta G_j(b) = (1 - G_j(b-))$. This eliminates the second and third terms. For any $a < b$, $\Delta G_j(a) = 0$, hence

$$E[S_i(b) - S_i(a) | \mathcal{F}_a] = (1 - G_j(a-))(U_{\bar{L}}(a) - L_a) \geq 0, \quad (\text{B.10})$$

with equality whenever $dG_i(a) = dG_j(a) > 0$. This proves (B.8).

Let now $a = \tau_i^{G, \vartheta}(1)$ and $b = \tau$ any stopping time taking values in $(\tau_i^{G, \vartheta}(1), \infty]$. It remains to show that $E[S_i(b) - S_i(a) | \mathcal{F}_a] \leq 0$ a.s. If $a = \tau_j^{G, \vartheta}(1)$, then $S_i(b) - S_i(a) = \Delta G_j(a)(F_a - M_a)$ and $\Delta G_j(a) > 0$ at $a = \tau_i^{G, \vartheta}(1) = \tau_j^{G, \vartheta}(1)$ only if $a = \tau^\vartheta$, whence $S_i(b) - S_i(a) = \Delta G_i(a)(F_a - M_a) \leq 0$ by hypothesis. If $a < \tau_j^{G, \vartheta}(1)$, we have $0 \leq (F - L)dG_j = (1 - G_j)\mathbf{1}_{F > L}dD_{\tilde{L}} \leq (1 - G_j)dD_{\tilde{L}}$ on $[a, b)$. Together with $\Delta G_j(b)M_b \leq \Delta G_j(b)\max(F_b, M_b)$ we get “ \leq ” in (B.9). On $\{b < \tau^\vartheta\}$ we now have $L_b = \tilde{L}_b \leq U_{\tilde{L}}(b)$ and $\Delta G_j(b) = 0$, so that “ \leq ” also replaces the equality in (B.10). As now $a = \tau_i^{G, \vartheta}(1) < \tau_j^{G, \vartheta}(1)$, $dD_{\tilde{L}}(a) > 0$ and $U_{\tilde{L}}(a) = L_a$. This finishes our proof. \square

Remark B.1. Theorem 5.1 remains true if L is only upper-semi-continuous from the right (and the left), but $L \equiv M$. Then $D_{\tilde{L}}$ will be left-continuous (see footnote 24) and there exists a feasible strategy G_i^ϑ given by

$$G_i^\vartheta(t) := 1 - \exp\left\{-\int_{\vartheta}^t \frac{dD_{\tilde{L}}^c(s)}{F_s - L_s} - \sum_{[\vartheta, u]} \ln\left(\frac{\Delta D_{\tilde{L}}(s)}{F_s - L_s} + 1\right)\right\}$$

for $t \in [\vartheta, \tau^\vartheta)$, where $D_{\tilde{L}}^c$ is the continuous part of $D_{\tilde{L}}$ and $\Delta D_{\tilde{L}}(s) = D_{\tilde{L}}(s+) - D_{\tilde{L}}(s)$, and which satisfies

$$dG_i^\vartheta(s) = (1 - G_i^\vartheta(s)) \frac{dD_{\tilde{L}}(s+)}{F_s - L_s}.$$

Then the only modifications in the proof are that we put $dD_{\tilde{L}}(\cdot +)$ on the RHS of (B.3), resp. the LHS (only!) of (B.4). We do not have right-continuity of $U_{\tilde{L}} - \tilde{L}$, but $\int_{[a, b)} (U_{\tilde{L}} - \tilde{L}) dG_i = 0$ still holds in (B.5): The argument of footnote 28 applies to $\Delta D_{\tilde{L}}$, which has the same support as ΔG_i . The continuous part dG_i^c is absolutely continuous with respect to $dD_{\tilde{L}}$, for which we can apply a change of variable similar to Lemma A.2, but with $\tau^{D_{\tilde{L}}}(x) := \inf\{t \geq 0 \mid D_{\tilde{L}}(t) > x\}$,⁴¹ such that $U_{\tilde{L}} = \tilde{L}$ a.s. at $\tau^{D_{\tilde{L}}}(x)$, $x \in \mathbb{R}_+$, cf. (3.4). Hence, $E[\int_{[0, \infty)} \mathbf{1}_{U_{\tilde{L}} > \tilde{L}} dD_{\tilde{L}}] = \int_0^\infty E[\mathbf{1}_{U_{\tilde{L}}(\tau^{D_{\tilde{L}}}(x)) > \tilde{L}(\tau^{D_{\tilde{L}}}(x))}] dx = 0$. Finally we employ $L = M$ to arrive at (B.6) if $\Delta G(a) > 0$.

Proof of Theorem 5.3. We only need to establish time consistency. If the hypothesis holds, $\{(\vartheta \vee \vartheta') \leq (\tau^\vartheta \wedge \tau^{\vartheta'})\}$ differs from $\{(\vartheta \vee \vartheta') \leq \tau^\vartheta = \tau^{\vartheta'}\} := A \in \mathcal{F}_{(\vartheta \vee \vartheta')}$ at most by a nullset. Only this event is relevant for time consistency, since $(\vartheta \vee \vartheta') > (\tau^\vartheta \wedge \tau^{\vartheta'})$ a.s. on A^c and there is no restriction where $G_i^\vartheta \vee G_i^{\vartheta'} = 1$. With $\tilde{L}^{\tau^\vartheta} = \tilde{L}^{\tau^{\vartheta'}}$ a.s. on A , also

$$\text{ess sup}_{\tau \leq \tau' \in \mathcal{T}} E[\tilde{L}^{\tau^\vartheta}(\tau') \mid \mathcal{F}_\tau] = \text{ess sup}_{\tau \leq \tau' \in \mathcal{T}} E[\tilde{L}^{\tau^{\vartheta'}}(\tau') \mid \mathcal{F}_\tau] \quad \text{a.s.}$$

on $\{\tau \geq (\vartheta \vee \vartheta')\} \cap A$ for any $\tau \in \mathcal{T}$, implying $U_{\tilde{L}}^{\tau^\vartheta} \mathbf{1}_{t \geq (\vartheta \vee \vartheta')} = U_{\tilde{L}}^{\tau^{\vartheta'}} \mathbf{1}_{t \geq (\vartheta \vee \vartheta')}$ a.s. on A (i.e., the latter two processes are indistinguishable) by the uniqueness of optional projec-

⁴¹For (left-) continuous $D_{\tilde{L}}$, $\tau^{D_{\tilde{L}}}(x) < t \Leftrightarrow D_{\tilde{L}}(t) > x$.

tions. Correspondingly, $D_L^{\tau^\vartheta} = D_L^{\tau^{\vartheta'}}$ on $[\vartheta \vee \vartheta', \infty]$ a.s. by the uniqueness of the Doob-Meyer decomposition.

The time consistency condition in the equivalent form $(1 - G_i^\vartheta(t)) = (1 - G_i^\vartheta(\vartheta' -))(1 - G_i^{\vartheta'}(t))$ reads for the given G_i^ϑ

$$\begin{aligned} & \mathbf{1}_{t < \tau^\vartheta} \exp\left(-\int_{\vartheta}^t dD_L^{\tau^\vartheta} / (F - L)\right) \\ &= \mathbf{1}_{\vartheta' \leq \tau^\vartheta} \exp\left(-\int_{\vartheta}^{\vartheta'} dD_L^{\tau^\vartheta} / (F - L)\right) \mathbf{1}_{t < \tau^{\vartheta'}} \exp\left(-\int_{\vartheta'}^t dD_L^{\tau^{\vartheta'}} / (F - L)\right), \end{aligned}$$

which on A and for $t \geq (\vartheta \vee \vartheta')$ reduces to the true statement

$$\mathbf{1}_{t < \tau^\vartheta} \exp\left(-\int_{\vartheta}^t dD_L^{\tau^\vartheta} / (F - L)\right) = \mathbf{1}_{t < \tau^\vartheta} \exp\left(-\int_{\vartheta}^{\vartheta'} dD_L^{\tau^\vartheta} / (F - L)\right) \exp\left(-\int_{\vartheta'}^t dD_L^{\tau^\vartheta} / (F - L)\right)$$

thanks to what we have shown before. The argument for j is analogous. \square

Proof of Theorem 7.3. Consider the subgame starting at a given $\vartheta \in \mathcal{T}$ and let $(G_1^\vartheta, \alpha_1^\vartheta)$ and $(G_2^\vartheta, \alpha_2^\vartheta)$ be a pair as hypothesized. First note that $\tau^\vartheta \leq \inf\{t \geq \vartheta \mid \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0\} = \hat{\tau}^\vartheta$ (cf. Definition C.1), such that $G_1^\vartheta = G_2^\vartheta = 1$ a.s. on $[\hat{\tau}^\vartheta, \infty]$. All other feasibility conditions for the extended mixed strategies follow from those of Theorem 5.1 and Proposition 7.1.

Now let $i, j \in \{1, 2\}$, $i \neq j$ arbitrary in the following (not necessarily the roles assigned in the theorem) and consider player i deviating to some admissible $(G_a^\vartheta, \alpha_a^\vartheta)$. G_j^ϑ is continuous up to $\tau_j := \tau_j^{G, \vartheta}(1) = \inf\{t \geq \vartheta \mid G_j^\vartheta(t) \geq 1\}$, whence only G_a^ϑ matters on $[\vartheta, \tau_j)$ (cf. Definition C.1):

$$V_i^\vartheta(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) = V_i^\vartheta(G_a^\vartheta, \mathbf{1}_{t \geq \tau_j} \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta).$$

So let $\alpha_a^\vartheta = 0$ on $[\vartheta, \tau_j)$ wlog. regarding optimality. Then $\tau_j \leq \inf\{t \geq \vartheta \mid \alpha_a^\vartheta(t) + \alpha_j^\vartheta(t) > 0\}$, implying

$$\begin{aligned} V_i^\vartheta(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) &= E \left[\int_{[0, \tau_j)} (1 - G_j^\vartheta) L dG_a^\vartheta + \int_{[0, \tau_j)} (1 - G_a^\vartheta) F dG_j^\vartheta + \sum_{[0, \tau_j)} M \Delta G_a^\vartheta \Delta G_j^\vartheta \right. \\ &\quad \left. + (1 - G_a^\vartheta(\tau_j -))(1 - G_j^\vartheta(\tau_j -)) V_i^{\tau_j}(G_a^{\tau_j}, \alpha_a^{\tau_j}, G_j^{\tau_j}, \alpha_j^{\tau_j}) \middle| \mathcal{F}_\vartheta \right] \quad (\text{B.11}) \end{aligned}$$

by iterated expectations, with $G_a^{\tau_j}$ and $G_j^{\tau_j}$ determined by time consistency (in particular, $G_a^{\tau_j}$ arbitrary where $G_a^{\tau_j}(\tau_j -) = 1$). Where G_j^ϑ jumps to 1 before τ^ϑ , by construction $F_{\tau_j} \geq M_{\tau_j}$, so waiting is a best reply, e.g. playing $G_a^{\tau_j} := \mathbf{1}_{t \geq \tau^\vartheta} G_a^\vartheta$ and $\alpha_a^{\tau_j} := \mathbf{1}_{t \geq \tau^\vartheta} \alpha_a^\vartheta$. Therefore

$$V_i^{\tau_j}(G_a^{\tau_j}, \alpha_a^{\tau_j}, G_j^{\tau_j}, \alpha_j^{\tau_j}) \leq V_i^{\tau_j}(\mathbf{1}_{t \geq \tau^\vartheta} G_a^\vartheta, \mathbf{1}_{t \geq \tau^\vartheta} \alpha_a^\vartheta, G_j^{\tau_j}, \alpha_j^{\tau_j}). \quad (\text{B.12})$$

Pasting G_a^ϑ and $G_a^{\tau_j}$ by time consistency yields $\mathbf{1}_{t < \tau_j} G_a^\vartheta + \mathbf{1}_{\tau_j \leq t < \tau^\vartheta} G_a^\vartheta(\tau_j -) + \mathbf{1}_{t \geq \tau^\vartheta} G_a^\vartheta$, which

in conjunction with $\mathbf{1}_{t \geq \tau^\vartheta} \alpha_a^\vartheta$ is (weakly) better than $(G_a^\vartheta, \alpha_a^\vartheta)$ – by combining (B.12) and (B.11). We may thus even assume $\alpha_a^\vartheta = 0$ on $[\vartheta, \tau^\vartheta)$ wlog. regarding optimality.

On $\{\tau^\vartheta < \hat{\tau}^\vartheta\}$, $M_{\tau^\vartheta} \geq F_{\tau^\vartheta}$ by right-continuity, whence $(G_i^{\tau^\vartheta}, \alpha_i^{\tau^\vartheta}) = (\mathbf{1}_{t \geq \tau^\vartheta}, \alpha_i^\vartheta)$ and $(G_j^{\tau^\vartheta}, \alpha_j^{\tau^\vartheta}) = (\mathbf{1}_{t \geq \tau^\vartheta}, \alpha_j^\vartheta)$ are mutual best replies with payoffs $V_i^{\tau^\vartheta}(G_i^{\tau^\vartheta}, \alpha_i^{\tau^\vartheta}, G_j^{\tau^\vartheta}, \alpha_j^{\tau^\vartheta}) = M_{\tau^\vartheta} = \max(F_{\tau^\vartheta}, M_{\tau^\vartheta})$. Proposition 7.1 shows that $(G_i^{\tau^\vartheta}, \alpha_i^{\tau^\vartheta}) = (\mathbf{1}_{t \geq \tau^\vartheta}, \alpha_i^\vartheta)$ and $(G_j^{\tau^\vartheta}, \alpha_j^{\tau^\vartheta}) = (\mathbf{1}_{t \geq \tau^\vartheta}, \alpha_j^\vartheta)$ are mutual best replies on $\{\tau^\vartheta = \hat{\tau}^\vartheta\}$ with payoffs $V_i^{\tau^\vartheta}(G_i^{\tau^\vartheta}, \alpha_i^{\tau^\vartheta}, G_j^{\tau^\vartheta}, \alpha_j^{\tau^\vartheta}) = \max(F_{\tau^\vartheta}, M_{\tau^\vartheta})$. For $\{\vartheta = \tau^\vartheta\}$ this directly implies optimality, while for $\{\vartheta < \tau^\vartheta\}$ we obtain the estimate

$$V_i^\vartheta(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \leq V_i^\vartheta(\mathbf{1}_{t < \tau^\vartheta} G_a^\vartheta + \mathbf{1}_{t \geq \tau^\vartheta}, \mathbf{1}_{t \geq \tau^\vartheta} \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta)$$

by extending (B.11) to τ^ϑ with $\tau^\vartheta \leq \inf\{t \geq \vartheta \mid \alpha_a^\vartheta(t) + \alpha_j^\vartheta(t) > 0\}$ thanks to the previous steps.

In summary, this means that for player i it suffices to verify optimality of G_i^ϑ against G_j^ϑ as (standard) feasible mixed strategies if we use the payoffs

$$\begin{aligned} V_i^\vartheta(G_a^\vartheta, G_j^\vartheta) &= E \left[\int_{[0, \tau^\vartheta)} (1 - G_j^\vartheta) L dG_a^\vartheta + \int_{[0, \tau^\vartheta)} (1 - G_a^\vartheta) F dG_j^\vartheta + \sum_{[0, \tau^\vartheta)} M \Delta G_a^\vartheta \Delta G_j^\vartheta \right. \\ &\quad \left. + \underbrace{(1 - G_a^\vartheta(\tau^\vartheta -))(1 - G_j^\vartheta(\tau^\vartheta -))}_{=(1 - G_a^\vartheta(\tau^\vartheta) + \Delta G_a^\vartheta(\tau^\vartheta)) \Delta G_j^\vartheta(\tau^\vartheta)} \max(F_{\tau^\vartheta}, M_{\tau^\vartheta}) \middle| \mathcal{F}_\vartheta \right] \\ &= E \left[\int_{[0, \infty]} S_i^\vartheta dG_a^\vartheta \middle| \mathcal{F}_\vartheta \right] \end{aligned}$$

for any feasible G_a^ϑ , where we set $S_i^\vartheta \equiv \int_{[0, \tau^\vartheta)} F dG_j^\vartheta + (1 - G_j^\vartheta(\tau^\vartheta -)) \max(F_{\tau^\vartheta}, M_{\tau^\vartheta})$ on $[\tau^\vartheta, \infty]$, on which in particular $G_j^\vartheta \equiv 1$; analogously for player j . This is however equivalent to the setting of Theorem 5.1 with $\Delta G_i^\vartheta(F - M) = 0$ a.s. at τ^ϑ (note that $\Delta G_i^\vartheta(F - M) \geq 0$ at $\tau_i^{G, \vartheta}(1) = \inf\{t \in \mathbb{R}_+ \mid G_i^\vartheta(t) = 1\}$ since $\tau^\vartheta \leq \inf\{t \geq \vartheta \mid M_t > F_t\}$), which proves optimality.

Time consistency of G_1 and G_2 is obtained exactly as in Theorem 5.3, and holds trivially for α_1 and α_2 because α_i^ϑ in Proposition 7.1 does not depend on ϑ (except for the feasibility condition $\alpha_i^\vartheta = 0$ on $[0, \vartheta)$, of course).

Finally, if either $F_\tau = M_\tau$ or $\tau = \inf\{t > \tau \mid L_t > F_t\}$ when $L_\tau = F_\tau$, then the above is equivalent to the setting of Theorem 5.1 with the condition $\Delta G_i^\vartheta(F - M) = 0$ a.s. at $\tau_i^{G, \vartheta}(1) < \tau^\vartheta$. \square

B.2 Proof of Proposition 8.1

We begin with Lemmata B.2 to B.4, which establish some important necessary conditions for strategies and payoffs in payoff-symmetric equilibria. They are crucial for the subsequent proof of Proposition 8.1.

Lemma B.2. *In any payoff-symmetric equilibrium, for any $\vartheta \in \mathcal{T}$,*

$$\int_{[0,t)} \mathbf{1}_{L \neq F} (1 - G_{2-}^\vartheta) dG_1^\vartheta = \int_{[0,t)} \mathbf{1}_{L \neq F} (1 - G_{1-}^\vartheta) dG_2^\vartheta$$

for all $t \in \mathbb{R}_+$ a.s. (with G_{i-}^ϑ the left limit $G_i^\vartheta - \Delta G_i^\vartheta$) and hence

$$\mathbf{1}_{L \neq F} \frac{dG_1^\vartheta}{(1 - G_{1-}^\vartheta)} = \mathbf{1}_{L \neq F} \frac{dG_2^\vartheta}{(1 - G_{2-}^\vartheta)}, \quad (\text{B.13})$$

which is to be interpreted as “= 0” if $(1 - G_{1-}^\vartheta)(1 - G_{2-}^\vartheta) = 0$. Both representations also hold with the right limits $(1 - G_1^\vartheta)$, $(1 - G_2^\vartheta)$.

Proof. First consider any $\tau \in \mathcal{T}$ with $\vartheta \leq \tau \leq \hat{\tau}^\vartheta = \inf\{t \geq \vartheta \mid \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0\}$ a.s. Time consistency and iterated expectations imply for $i, j \in \{1, 2\}$, $i \neq j$,

$$\begin{aligned} V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) &= E \left[\int_{[0,\tau)} (1 - G_j^\vartheta) L dG_i^\vartheta + \int_{[0,\tau)} (1 - G_i^\vartheta) F dG_j^\vartheta + \sum_{[0,\tau)} M \Delta G_i^\vartheta \Delta G_j^\vartheta \right. \\ &\quad \left. + (1 - G_i^\vartheta(\tau-))(1 - G_j^\vartheta(\tau-)) V_i^\tau(G_i^\tau, \alpha_i^\tau, G_j^\tau, \alpha_j^\tau) \middle| \mathcal{F}_\vartheta \right] \\ &= E \left[\int_{[0,\tau)} (1 - G_{j-}^\vartheta) L dG_i^\vartheta + \int_{[0,\tau)} (1 - G_{i-}^\vartheta) F dG_j^\vartheta \right. \\ &\quad \left. + \sum_{[0,\tau)} (M - L - F) \Delta G_i^\vartheta \Delta G_j^\vartheta \right. \\ &\quad \left. + (1 - G_i^\vartheta(\tau-))(1 - G_j^\vartheta(\tau-)) V_i^\tau(G_i^\tau, \alpha_i^\tau, G_j^\tau, \alpha_j^\tau) \middle| \mathcal{F}_\vartheta \right]. \end{aligned} \quad (\text{B.14})$$

In a payoff-symmetric equilibrium $V_1^\vartheta - V_2^\vartheta = V_1^\tau - V_2^\tau = 0$, hence

$$E \left[\int_{[0,\tau)} (1 - G_{2-}^\vartheta) (L - F) dG_1^\vartheta - \int_{[0,\tau)} (1 - G_{1-}^\vartheta) (L - F) dG_2^\vartheta \middle| \mathcal{F}_\vartheta \right] = 0$$

for any $\tau \in [\vartheta, \hat{\tau}^\vartheta]$. The integrals represent two signed optional random measures,⁴² which agree on all optional sets in $[0, \hat{\tau}^\vartheta]$ (trivially on $[0, \vartheta)$ by $G_i^\vartheta(\vartheta-) = 0$; the optional σ -field is generated by the stochastic intervals $[0, \tau)$, $\tau \in \mathcal{T}$). L and F are optional processes, hence we may cancel $(L - F) \neq 0$ to observe two left-continuous (thus optional) processes $\int_{[0,t)} \mathbf{1}_{L \neq F} (1 - G_{2-}^\vartheta) dG_1^\vartheta$ and $\int_{[0,t)} \mathbf{1}_{L \neq F} (1 - G_{1-}^\vartheta) dG_2^\vartheta$ that agree in expectation at any stopping time $\tau \leq \hat{\tau}^\vartheta$. They are thus indistinguishable up to $\hat{\tau}^\vartheta$ by the uniqueness of optional projections.

The two measures $\int_{[0,t)} \mathbf{1}_{L \neq F} (1 - G_{j-}^\vartheta) dG_i^\vartheta$ do not charge $(\hat{\tau}^\vartheta, \infty)$, where $G_1^\vartheta \vee G_2^\vartheta \equiv 1$. For the remaining set $[\hat{\tau}^\vartheta]$, suppose $\hat{\tau}^\vartheta = \hat{\tau}_j^\vartheta$ (cf. Definition C.1), so $G_j^\vartheta(\hat{\tau}^\vartheta) = G_j^{\hat{\tau}^\vartheta}(\hat{\tau}^\vartheta) = 1$ by

⁴² $\int_{[0,t)} (1 - G_{2-}^\vartheta) (L - F) dG_1^\vartheta$ and $\int_{[0,t)} (1 - G_{1-}^\vartheta) (L - F) dG_2^\vartheta$ are adapted, right-continuous and of finite variation. Their minimal decomposition is using $(L - F)^+$ and $(L - F)^-$, respectively.

time consistency. Now, if $G_i^{\hat{\tau}^\vartheta}(\hat{\tau}^\vartheta) = 0$, $V_i^{\hat{\tau}^\vartheta} = F_{\hat{\tau}^\vartheta}$ and $V_j^{\hat{\tau}^\vartheta} = L_{\hat{\tau}^\vartheta}$, implying $L_{\hat{\tau}^\vartheta} = F_{\hat{\tau}^\vartheta}$.

If $G_i^{\hat{\tau}^\vartheta}(\hat{\tau}^\vartheta) \in (0, 1)$, then, by Riedel and Steg (2014), Section 3, i is indifferent between stopping and waiting, i.e., $V_i^{\hat{\tau}^\vartheta} = F_{\hat{\tau}^\vartheta} = \alpha_j^\vartheta(\hat{\tau}^\vartheta)M_{\hat{\tau}^\vartheta} + (1 - \alpha_j^\vartheta(\hat{\tau}^\vartheta))L_{\hat{\tau}^\vartheta}$. On the other hand, $V_j^{\hat{\tau}^\vartheta} = G_i^{\hat{\tau}^\vartheta}(\hat{\tau}^\vartheta)[\alpha_j^\vartheta(\hat{\tau}^\vartheta)M_{\hat{\tau}^\vartheta} + (1 - \alpha_j^\vartheta(\hat{\tau}^\vartheta))F_{\hat{\tau}^\vartheta}] + (1 - G_i^{\hat{\tau}^\vartheta}(\hat{\tau}^\vartheta))L_{\hat{\tau}^\vartheta}$. If $M_{\hat{\tau}^\vartheta} > F_{\hat{\tau}^\vartheta}$, j would set $\alpha_j^\vartheta(\hat{\tau}^\vartheta) = 1$, contradicting the indifference of i . If $M_{\hat{\tau}^\vartheta} < F_{\hat{\tau}^\vartheta}$, $\alpha_j^\vartheta(\hat{\tau}^\vartheta) = 0$ and hence $L_{\hat{\tau}^\vartheta} = F_{\hat{\tau}^\vartheta}$ by the indifference of i . Finally, if $M_{\hat{\tau}^\vartheta} = F_{\hat{\tau}^\vartheta}$, $V_i^{\hat{\tau}^\vartheta} = F_{\hat{\tau}^\vartheta} = V_j^{\hat{\tau}^\vartheta} = G_i^{\hat{\tau}^\vartheta}(\hat{\tau}^\vartheta)F_{\hat{\tau}^\vartheta} + (1 - G_i^{\hat{\tau}^\vartheta}(\hat{\tau}^\vartheta))L_{\hat{\tau}^\vartheta}$, and hence again $L_{\hat{\tau}^\vartheta} = F_{\hat{\tau}^\vartheta}$. In summary, we must have $G_i^{\hat{\tau}^\vartheta}(\hat{\tau}^\vartheta) = 1$ if $L_{\hat{\tau}^\vartheta} \neq F_{\hat{\tau}^\vartheta}$ and then by time consistency $\Delta G_i^\vartheta(\hat{\tau}^\vartheta) = 1 - G_i^\vartheta(\hat{\tau}^\vartheta -)$ and $\Delta G_j^\vartheta(\hat{\tau}^\vartheta) = 1 - G_j^\vartheta(\hat{\tau}^\vartheta -)$, which makes our measures above agree on all of $[0, \infty)$.

The representation (B.13) is obtained by integrating $[(1 - G_{1-}^\vartheta)(1 - G_{2-}^\vartheta)]^{-1}$ on $\{G_{1-}^\vartheta \vee G_{2-}^\vartheta < 1\}$ w.r.t. each measure. Finally, $\mathbf{1}_{L \neq F}(1 - G_2^\vartheta) dG_1^\vartheta = \mathbf{1}_{L \neq F}(1 - G_1^\vartheta) dG_2^\vartheta$ is obtained analogously, skipping the step (B.14). \square

Lemma B.3. $G_1^\vartheta = G_2^\vartheta$ on $[\vartheta, \inf\{t \geq \vartheta \mid \mathbf{1}_{L=F}(dG_1^\vartheta + dG_2^\vartheta) > 0\})$ a.s.

Proof. It is easier to work with $\mathbf{1}_{L \neq F}(1 - G_j^\vartheta) dG_i^\vartheta = \mathbf{1}_{L \neq F}(1 - G_i^\vartheta) dG_j^\vartheta$, $i, j \in \{1, 2\}$, $i \neq j$. On the interval $[\vartheta, \inf\{t \geq \vartheta \mid \mathbf{1}_{L=F}(dG_1^\vartheta + dG_2^\vartheta) > 0\})$ we can ignore $\mathbf{1}_{L \neq F}$, whence we now get $G_i^\vartheta(\vartheta) = G_j^\vartheta(\vartheta)$ and, while $G_j^\vartheta < 1$, $dG_i^\vartheta = \phi dG_j^\vartheta$ with

$$\phi = \frac{1 - G_i^\vartheta}{1 - G_j^\vartheta}, \quad \phi_0 = 1. \quad (\text{B.15})$$

By rearranging (B.15), for all $t \in [\vartheta, \inf\{t \geq \vartheta \mid \mathbf{1}_{L=F}(dG_1^\vartheta + dG_2^\vartheta) > 0\}) \cap \{G_j^\vartheta(t) < 1\}$,

$$\begin{aligned} G_i^\vartheta(t) &= 1 - (1 - G_j^\vartheta(t))\phi_t = G_i^\vartheta(0) + \int_{(0,t]} \phi dG_j^\vartheta \\ \Leftrightarrow \quad \phi_t - 1 &= G_j^\vartheta(t)\phi_t - G_j^\vartheta(0) - \int_{(0,t]} \phi dG_j^\vartheta \\ \Leftrightarrow \quad \phi_t - \phi_0 &= \int_{(0,t]} G_{j-}^\vartheta d\phi. \end{aligned}$$

The last line is obtained by integration by parts, as ϕ is right-continuous while $G_j^\vartheta < 1$. It implies that ϕ must indeed be constant where $G_j^\vartheta < 1$. If G_j^ϑ jumps to 1 where $L \neq F$ then $(1 - G_i^\vartheta)\Delta G_j^\vartheta = (1 - G_j^\vartheta)\Delta G_i^\vartheta = 0$, i.e., G_i^ϑ must attain 1, too, which completes the proof. \square

Lemma B.4. Suppose $\Delta G_1^\vartheta(\vartheta) \vee \Delta G_2^\vartheta(\vartheta) > 0$ in a payoff-symmetric equilibrium. Then

$$F_\vartheta \leq \max(L_\vartheta, M_\vartheta) \quad \text{and} \quad V_1^\vartheta = V_2^\vartheta \leq \Delta G_j^\vartheta(\vartheta) \max(F_\vartheta, M_\vartheta) + (1 - \Delta G_j^\vartheta(\vartheta))L_\vartheta$$

for $j = 1, 2$, a.s.

Proof. To simplify notation, note that $\Delta G_i^\vartheta(\vartheta) = G_i^\vartheta(\vartheta)$, $i = 1, 2$. First consider $\vartheta < \hat{\tau}^\vartheta$. If

$G_1^\vartheta(\vartheta) \wedge G_2^\vartheta(\vartheta) > 0$, either player i is willing to stop, so $V_i^\vartheta = G_j^\vartheta(\vartheta)M_\vartheta + (1 - G_j^\vartheta(\vartheta))L_\vartheta \geq G_j^\vartheta(\vartheta)F_\vartheta + (1 - G_j^\vartheta(\vartheta))L_\vartheta$ and hence $M_\vartheta \geq F_\vartheta$, a.s., proving the claim.⁴³

If $G_1^\vartheta(\vartheta) \wedge G_2^\vartheta(\vartheta) = 0$, then Lemma B.2 (resp. equation (B.13)) implies $L_\vartheta = F_\vartheta$ a.s. The player i with $G_i^\vartheta(\vartheta) > 0$ is willing to stop, so $V_i^\vartheta = L_\vartheta = G_i^\vartheta(\vartheta)F_\vartheta + (1 - G_i^\vartheta(\vartheta))L_\vartheta = V_j^\vartheta \geq G_i^\vartheta(\vartheta)M_\vartheta + (1 - G_i^\vartheta(\vartheta))L_\vartheta$. The second equality just applies $L_\vartheta = F_\vartheta$ and the estimate is player j 's opportunity to stop (it now shows also $F_\vartheta \geq M_\vartheta$, a.s.).

With extended mixed strategies, suppose now $\vartheta = \hat{\tau}_j^\vartheta$ (cf. Definition C.1), so $G_j^\vartheta(\vartheta) = 1$. Then, by Riedel and Steg (2014), Section 3, $V_i^\vartheta = \max\{F_\vartheta, \alpha_j^\vartheta(\vartheta)M_\vartheta + (1 - \alpha_j^\vartheta(\vartheta))L_\vartheta\}$. On $\{F_\vartheta > \max(M_\vartheta, L_\vartheta)\}$, player i would ensure to become follower to realize $V_i^\vartheta = F_\vartheta$, whence j would become leader for sure with $V_j^\vartheta = L_\vartheta < F_\vartheta$, contradicting payoff symmetry.

Now suppose $V_i^\vartheta > \max(F_\vartheta, M_\vartheta)$. Then we would have $V_i^\vartheta = \alpha_j^\vartheta(\vartheta)M_\vartheta + (1 - \alpha_j^\vartheta(\vartheta))L_\vartheta > F_\vartheta$, and i would ensure *not* to become follower. Then j would for sure not become leader and $V_j^\vartheta \leq \max(F_\vartheta, M_\vartheta)$, contradicting payoff symmetry. This proves the claim for i . If also $\vartheta = \hat{\tau}_i^\vartheta$ and hence $G_i^\vartheta(\vartheta) = 1$, reversing roles shows that $V_j^\vartheta \leq \max(F_\vartheta, M_\vartheta)$. If $\vartheta = \hat{\tau}_j^\vartheta < \hat{\tau}_i^\vartheta$, $V_j^\vartheta = G_i^\vartheta(\vartheta)[\alpha_j^\vartheta(\vartheta)M_\vartheta + (1 - \alpha_j^\vartheta(\vartheta))F_\vartheta] + (1 - G_i^\vartheta(\vartheta))L_\vartheta \leq G_i^\vartheta(\vartheta)\max(F_\vartheta, M_\vartheta) + (1 - G_i^\vartheta(\vartheta))L_\vartheta$, finishing the proof. \square

Proof of Proposition 8.1. Fix $\vartheta \in \mathcal{T}$. By optimality we must have for any $\tau \geq \vartheta$ with $G_i^\vartheta(\tau-) < 1$ a.s. that

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) = V_i^\vartheta(G_i^\tau, \alpha_i^\tau, G_j^\tau, \alpha_j^\tau)$$

in any equilibrium. By time consistency and iterated expectations,

$$V_i^\vartheta(G_i^\tau, \alpha_i^\tau, G_j^\tau, \alpha_j^\tau) = E \left[\int_{[0, \tau)} F dG_j^\vartheta + (1 - G_j^\vartheta(\tau-))V_i^\tau(G_i^\tau, \alpha_i^\tau, G_j^\tau, \alpha_j^\tau) \middle| \mathcal{F}_\vartheta \right]. \quad (\text{B.16})$$

Now define the stopping times

$$\hat{\tau}(x) := \inf\{t \geq \vartheta \mid \mathbf{1}_{F \geq L}(dG_i^\vartheta + dG_j^\vartheta) > 0\} \wedge \inf\{t \geq \vartheta \mid G_i^\vartheta \vee G_j^\vartheta > x\}$$

for $x \in [0, 1)$, and their monotone limit $\hat{\tau}(1) := \lim_{x \nearrow 1} \hat{\tau}(x)$. Then (B.16) holds for all $\hat{\tau}(x)$, $0 \leq x < 1$, with $F dG_j^\vartheta = (L \wedge F) dG_j^\vartheta$. The integral converges as $x \rightarrow 1$, also in expectation because F is of class (D).

First consider the set $\{G_i^\vartheta(\hat{\tau}(1)-) \vee G_j^\vartheta(\hat{\tau}(1)-) < 1\}$, where $V_i^{\hat{\tau}(x)} \rightarrow V_i^{\hat{\tau}(1)}$ a.s. Further, if $G_i^\vartheta \vee G_j^\vartheta < 1$ at $\hat{\tau}(1)$, then $\mathbf{1}_{F \geq L}(dG_i^\vartheta + dG_j^\vartheta) > 0$, hence $F_{\hat{\tau}(1)} \geq L_{\hat{\tau}(1)}$ and $\hat{\tau}(1)$ is a point of increase for either G_i^ϑ or G_j^ϑ , so at least one player is willing to stop. If there is no jump, then $V_i^{\hat{\tau}(1)} = V_j^{\hat{\tau}(1)} = L_{\hat{\tau}(1)}$. If there is any jump, then $F_{\hat{\tau}(1)} = L_{\hat{\tau}(1)}$ and $V_i^{\hat{\tau}(1)} \leq F_{\hat{\tau}(1)}$ by

⁴³Suppose that $G_i^\vartheta(\tau^*) > 0$ in an equilibrium, $\hat{\tau}^\vartheta > \tau^* \in \mathcal{T}$, so $E[S_i^\vartheta(\tau^*) \mid \mathcal{F}_\vartheta] = \text{ess sup}_{\vartheta \leq \tau \in \mathcal{T}} E[S_i^\vartheta(\tau) \mid \mathcal{F}_\vartheta] \geq \lim_{\varepsilon \searrow 0} E[S_i^\vartheta(\tau^* + \varepsilon) \mid \mathcal{F}_\vartheta]$. As L is right-continuous by assumption, we have $S_i^\vartheta(t+) - S_i^\vartheta(t) = (F_t - M_t)\Delta G_j^\vartheta(t)$. Since S_i^ϑ is of class (D), $\lim_{\varepsilon \searrow 0} E[S_i^\vartheta(\tau^* + \varepsilon) \mid \mathcal{F}_\vartheta] = E[S_i^\vartheta(\tau^*+) \mid \mathcal{F}_\vartheta]$, hence $(F_{\tau^*} - M_{\tau^*})\Delta G_j^\vartheta(\tau^*) \leq 0$ a.s.

Lemma B.4 and the hypothesis $M \leq F$. On the other hand, if $G_i^\vartheta \vee G_j^\vartheta = 1$ at $\hat{\tau}(1)$, then $G_i^{\hat{\tau}(1)}(\hat{\tau}(1)) \vee G_j^{\hat{\tau}(1)}(\hat{\tau}(1)) = 1$, hence $V_i^{\hat{\tau}(1)} \leq F_{\hat{\tau}(1)} \leq L_{\hat{\tau}(1)}$ by Lemma B.4 and $M \leq \min(F, L)$.

Now consider the set $\{G_i^\vartheta(\hat{\tau}(1)-) \vee G_j^\vartheta(\hat{\tau}(1)-) = 1\}$, which means that one distribution has fully grown on $\{F < L\}$. Then in fact $G_i^\vartheta(\hat{\tau}(1)-) = G_j^\vartheta(\hat{\tau}(1)-) = 1$ by Lemma B.3. Thus, $(1 - G_j^\vartheta(\hat{\tau}(x)-))V_i^{\hat{\tau}(x)} \rightarrow 0$ a.s. because $(V_i^{\hat{\tau}(x)})_{x \in [0,1]}$ is uniformly integrable (see Lemma A.7).

In total, we have that $(1 - G_j^\vartheta(\hat{\tau}(x)-))V_i^{\hat{\tau}(x)}$ in (B.16) converges a.s. and in expectation to a limit bounded by $(1 - G_j^\vartheta(\hat{\tau}(1)-))(L_{\hat{\tau}(1)} \wedge F_{\hat{\tau}(1)})$, which completes the proof. Note that we can also state the result in the form

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \leq \Delta G_j^\vartheta(\vartheta)F_\vartheta + (1 - \Delta G_j^\vartheta(\vartheta))U_{L \wedge F}(\vartheta) \leq U_{L \wedge F}(\vartheta). \quad \square$$

B.3 Proof of Theorem 8.2

Proposition 8.1 shows that in a payoff-symmetric equilibrium, the continuation values are at most

$$\Delta G_j^\vartheta(\vartheta)F_\vartheta + (1 - \Delta G_j^\vartheta(\vartheta)) \operatorname{ess\,sup}_{\vartheta \leq \tau \in \mathcal{T}} E[L_\tau \wedge F_\tau \mid \mathcal{F}_\vartheta] \leq U_{L \wedge F}(\vartheta),$$

in fact considering only stopping times $\tau \leq \inf\{t \geq \vartheta \mid G_i^\vartheta \vee G_j^\vartheta = 1\}$. However, if $L > U_{L \wedge F}$ and $G_j^\vartheta < 1$, player i 's value V_i^ϑ would be at least $\Delta G_j^\vartheta(\vartheta)F_\vartheta + (1 - \Delta G_j^\vartheta(\vartheta))L_\vartheta$ by considering the limit of stopping at $\vartheta + \varepsilon$ with $\varepsilon \searrow 0$ (for convergence, cf. footnote 43); a contradiction. Therefore the game beginning at ϑ cannot continue past $\tau_0(\vartheta)$, and the continuation values are actually bounded by $U_{(L \wedge F)\tau_0(\vartheta)} \leq U_{L \wedge F}$. This argument of course iterates, inducing decreasing sequences of stopping times $(\tau_n(\vartheta))_n$ and Snell envelopes $(U_{(L \wedge F)\tau_n(\vartheta)})_n$, as one introduces increasingly strict constraints. The decreasing sequence of stopping times is bounded below by ϑ , so the limit $\tilde{\tau}(\vartheta) = \lim_{n \rightarrow \infty} \tau_n(\vartheta)$ exists and is a stopping time as well.

It is convenient to have optimal stopping times that attain $U_{(L \wedge F)\tau_n(\vartheta)}$, respectively. They exist because each process $(L \wedge F)^{\tau_n(\vartheta)}$ is clearly right-continuous and of class (D), but also upper-semi-continuous in expectation under Assumption 2.1 (iii).

These optimal stopping times simplify the argument to prove $\tilde{\tau}(\vartheta) \leq \inf\{t \geq \vartheta \mid L > U_{(L \wedge F)\tilde{\tau}(\vartheta)}\}$. Suppose to the contrary that there is a stopping time $\sigma \in [\vartheta, \tilde{\tau}(\vartheta)]$ with $L_\sigma > U_{(L \wedge F)\tilde{\tau}(\vartheta)}(\sigma)$ with positive probability, i.e., $L_\sigma > E[L_\tau \wedge F_\tau \mid \mathcal{F}_\sigma]$ for all $\tau \in [\sigma, \tilde{\tau}(\vartheta)]$. Since $L_\sigma \leq U_{(L \wedge F)\tau_n(\vartheta)}(\sigma)$ a.s. for each $n \in \mathbb{N}$, there must exist stopping times $\sigma_n \in [\tilde{\tau}(\vartheta), \tau_n(\vartheta)]$ such that $L_\sigma \leq E[L_{\sigma_n} \wedge F_{\sigma_n} \mid \mathcal{F}_\sigma]$. Necessarily $\sigma_n \rightarrow \tilde{\tau}(\vartheta)$ a.s., and by right-continuity and L being of class (D), $L_\sigma \leq E[L_{\tilde{\tau}(\vartheta)} \wedge F_{\tilde{\tau}(\vartheta)} \mid \mathcal{F}_\sigma] \leq U_{(L \wedge F)\tilde{\tau}(\vartheta)}(\sigma)$. Thus there is no threat of preemption on $[\vartheta, \tilde{\tau}(\vartheta))$ if $U_{(L \wedge F)\tilde{\tau}(\vartheta)}$ is the continuation value.

$\tilde{\tau}(\vartheta)$ is indeed maximal by construction. Whenever we have another $\hat{\tau} \geq \vartheta$ such that $L \leq U_{(L \wedge F)\hat{\tau}}$ on $[\vartheta, \hat{\tau})$, then $\hat{\tau} \leq \tau_0(\vartheta)$ a.s., because $U_{(L \wedge F)\hat{\tau}} \leq U_{L \wedge F}$. So $U_{(L \wedge F)\hat{\tau}} \leq U_{(L \wedge F)\tau_0(\vartheta)}$ and by iteration $\hat{\tau} \leq \tau_n(\vartheta)$, $n \in \mathbb{N}$.

Let us remark some further regularity properties. Since $U_{(L \wedge F)^{\tau_n(\vartheta)}} \geq (L \wedge F)^{\tau_n(\vartheta)}$, we can only have $L > U_{(L \wedge F)^{\tau_n(\vartheta)}}$ if $L > F$, thus $L \geq F$ by right-continuity at all $\tau_k(\vartheta)$, $k \in \mathbb{N}_0$, and at $\tilde{\tau}(\vartheta)$, too. Moreover,

$$\tilde{\tau}(\vartheta) = \inf\{t \geq \tilde{\tau}(\vartheta) \mid L_t > F_t\} \quad \text{a.s.} \quad (\text{B.17})$$

because we would otherwise have $L = L \wedge F \leq U_{(L \wedge F)^{\tau_k(\vartheta)}}$ for all $k \in \mathbb{N}_0$ until the stopping time on the right. By construction we have $\tilde{\tau}(\vartheta') = \tilde{\tau}(\vartheta)$ a.s. on $\{\vartheta' \in [\vartheta, \tilde{\tau}(\vartheta)]\}$ for any $\vartheta' \in \mathcal{T}$, hence $\tilde{\tau}(\vartheta)$ is monotone in ϑ and

$$\tilde{\tau}(\vartheta) = \lim_{n \rightarrow \infty} \tilde{\tau}(\vartheta_n) \quad \text{a.s.} \quad (\text{B.18})$$

for any sequence $\vartheta_n \searrow \vartheta$, $(\vartheta_n)_n \subset \mathcal{T}$. Indeed, (B.18) also holds on $\{\vartheta = \tilde{\tau}(\vartheta)\}$: If we denote the limit by $\tilde{\tau}' \geq \tilde{\tau}(\vartheta)$, then $\tilde{\tau}(\vartheta_n) = \tilde{\tau}'$ on $\{\vartheta_n \leq \tilde{\tau}'\}$, and therefore, using that $U_{(L \wedge F)^{\tilde{\tau}'}}$ is a supermartingale,

$$\begin{aligned} U_{(L \wedge F)^{\tilde{\tau}'}}(\vartheta) &\geq E[U_{(L \wedge F)^{\tilde{\tau}'}}(\vartheta_n) \mid \mathcal{F}_\vartheta] \\ &\geq E[L_{\vartheta_n} \mathbf{1}_{\vartheta_n < \tilde{\tau}'} + F_{\tilde{\tau}'} \mathbf{1}_{\vartheta_n \geq \tilde{\tau}'} \mid \mathcal{F}_\vartheta] \end{aligned}$$

for any $n \in \mathbb{N}$. The term on the second line goes to L_ϑ on $\{\vartheta < \tilde{\tau}'\}$ as $n \rightarrow \infty$, thanks to L being right-continuous and of class (D). This argument implies $L \leq U_{(L \wedge F)^{\tilde{\tau}'}}$ on $[\vartheta, \tilde{\tau}')$. By the maximality of $\tilde{\tau}(\vartheta)$ shown before, $\tilde{\tau}(\vartheta) = \tilde{\tau}'$ a.s.

Now define the process \tilde{L} by

$$\tilde{L}_t := \begin{cases} L_t & t < \tilde{\tau}(\vartheta) \\ F_{\tilde{\tau}(\vartheta)} & t \geq \tilde{\tau}(\vartheta) \end{cases}$$

with Snell envelope $U_{\tilde{L}}$ and its compensator $D_{\tilde{L}}$. From $L_{\tilde{\tau}(\vartheta)} \geq F_{\tilde{\tau}(\vartheta)}$ we see that $(L \wedge F)^{\tilde{\tau}(\vartheta)} \leq \tilde{L} \leq U_{(L \wedge F)^{\tilde{\tau}(\vartheta)}} \leq U_{\tilde{L}}$ on $[\vartheta, \infty]$. Since the Snell envelope is the smallest supermartingale dominating the payoff process, the last inequality must actually bind and thus also $D_{\tilde{L}} = D_{(L \wedge F)^{\tilde{\tau}(\vartheta)}}$ on $[\vartheta, \infty]$. By this observation we obtain

$$\int_{[\vartheta, \infty]} \mathbf{1}_{L > F} dD_{\tilde{L}} = \sum_{[\vartheta, \infty]} \Delta D_{\tilde{L}} = 0 \quad \text{a.s.}, \quad (\text{B.19})$$

which we need to apply Theorem 5.1 for ϑ and $\tau^\vartheta = \tilde{\tau}(\vartheta)$. Indeed, the path regularity of $(L \wedge F)^{\tilde{\tau}(\vartheta)}$ verified before implies continuity of $D_{(L \wedge F)^{\tilde{\tau}(\vartheta)}}$ and hence that $L \leq U_{(L \wedge F)^{\tilde{\tau}(\vartheta)}} = L \wedge F$, i.e., $F \geq L$ at any point of increase of $dD_{(L \wedge F)^{\tilde{\tau}(\vartheta)}}$ on $[\vartheta, \tilde{\tau}(\vartheta))$. Furthermore, $D_{(L \wedge F)^{\tilde{\tau}(\vartheta)}}$ is right-continuous and flat from $\tilde{\tau}(\vartheta)$ on, so it does not charge $[\tilde{\tau}(\vartheta), \infty]$, a.s.

With (B.19) and (B.17) we can now define supporting equilibrium strategies $(\tilde{G}_1^\sigma, \tilde{\alpha}_1^\sigma)_{\sigma \in \mathcal{T}}$ and $(\tilde{G}_2^\sigma, \tilde{\alpha}_2^\sigma)_{\sigma \in \mathcal{T}}$ as follows. First, if we concentrate on a fixed ϑ , it is enough to set, for any $\sigma \in \mathcal{T}$,

$$\tilde{\alpha}_i^\sigma := (1 - \mathbf{1}_{[\vartheta, \tilde{\tau}(\vartheta))}) \alpha_i^\sigma$$

with α_i^σ as in Proposition 7.1, $i = 1, 2$. Now pick $i, j \in \{1, 2\}$, $i \neq j$, and define $\tilde{G}_i^\sigma := G_i^\sigma$, $\tilde{G}_j^\sigma := G_j^\sigma$ as in Theorem 5.1 with $\tau^\sigma := \inf\{t \geq \sigma \mid (1 - \mathbf{1}_{[\vartheta, \tilde{\tau}(\vartheta))})(L_t - F_t) > 0\}$ ($= \tilde{\tau}(\vartheta)$ a.s. on $\{\sigma \in [\vartheta, \tilde{\tau}(\vartheta)]\}$ by (B.17)). These strategies form a subgame-perfect equilibrium analogously to Theorem 7.3 with payoff $U_{(L \wedge F)^{\tilde{\tau}(\vartheta)}}(\vartheta') = U_{\tilde{L}}(\vartheta')$ at any $\vartheta' \in [\vartheta, \tilde{\tau}(\vartheta)]$, since $\tilde{\tau}(\vartheta') = \tilde{\tau}(\vartheta)$ a.s. (note that the endpoint condition $\Delta G_i^\vartheta(F - M) \geq 0$ at $\tau_i^{G, \vartheta}(1) = \inf\{t \in \mathbb{R}_+ \mid G_i^\vartheta(t) = 1\}$ is void thanks to $M \leq F$).

If we want the respective maximal payoff to be attained in *every* subgame in one equilibrium, we need to be careful how to aggregate the random intervals $[\vartheta, \tilde{\tau}(\vartheta))$ – on which to suppress α_i – in a measurable way across all $\vartheta \in \mathcal{T}$. Therefore, begin with the (optional) set $A := \bigcup_{q \in \mathbb{Q}_+} [q, \tilde{\tau}(q))$. Now approximate any given $\vartheta \in \mathcal{T}$ from above by the decreasing sequence $(\vartheta_n)_{n \in \mathbb{N}}$, where $\vartheta_n := 2^{-n} \lceil 2^n \vartheta \rceil \in \mathcal{T}$. Any ϑ_n takes values only in $\{k2^{-n}; k \in \bar{\mathbb{N}}\}$ and thus $[\vartheta_n, \tilde{\tau}(\vartheta_n)) = \sum_{k \in \bar{\mathbb{N}}} [k2^{-n}, \tilde{\tau}(k2^{-n})) \mathbf{1}_{\vartheta_n = k2^{-n}}$ a.s., which means that $[\vartheta_n, \tilde{\tau}(\vartheta_n))$ belongs to A up to a nullset. Further, $\vartheta_n \leq \vartheta + 2^{-n}$ implies $\tilde{\tau}(\vartheta) = \tilde{\tau}(\vartheta_k)$ for all $k \geq n$ on $\{\tilde{\tau}(\vartheta) \geq \vartheta + 2^{-n}\}$ a.s., and therefore on this set $(\vartheta, \tilde{\tau}(\vartheta)) = \bigcup_{k \geq n} [\vartheta_k, \tilde{\tau}(\vartheta_k))$ a.s., which means that $(\vartheta, \tilde{\tau}(\vartheta))$ belongs to A on $\{\tilde{\tau}(\vartheta) \geq \vartheta + 2^{-n}\}$ up to a nullset. Aggregating, $(\vartheta, \tilde{\tau}(\vartheta)) = (\vartheta, \tilde{\tau}(\vartheta))(\mathbf{1}_{\tilde{\tau}(\vartheta) \geq 2^{-1}} + \sum_n \mathbf{1}_{\tilde{\tau}(\vartheta) \in [\vartheta + 2^{-n-1}, \vartheta + 2^{-n})})$ a.s. and hence belongs to A up to a nullset.

Now define

$$\tilde{\alpha}_i^\vartheta := \left(\limsup_{u \searrow \vartheta} \mathbf{1}_{u \in A^c} \right) \alpha_i^\vartheta$$

for any $\vartheta \in \mathcal{T}$, with α_i^ϑ as in Proposition 7.1, $i = 1, 2$. $\tilde{\alpha}_i^\vartheta$ does not depend on ϑ and is still progressively measurable by Theorem IV.33 (c) in Dellacherie and Meyer (1978). Further, $\tilde{\alpha}_i^\vartheta = 0$ on $[\vartheta, \tilde{\tau}(\vartheta))$ a.s. It remains to verify right-continuity (where $\tilde{\alpha}_i^\vartheta < 1$) for a subgame-perfect equilibrium along the previous lines, and that $\tilde{\tau}(\vartheta) = \inf\{t \geq \tilde{\tau}(\vartheta) \mid \tilde{\alpha}_i^\vartheta(t) > 0\}$ a.s.

Consider the set $\{\limsup_{u \searrow \sigma} \mathbf{1}_{u \in A^c} \neq \liminf_{u \searrow \sigma} \mathbf{1}_{u \in A^c}\}$ for some $\sigma \in \mathcal{T}$. We cannot have $L_\sigma < F_\sigma$ on that set with positive probability, because then we would have $\tilde{\tau}(\sigma) \geq \inf\{t \geq \sigma \mid F_t > L_t\} > \sigma$ and $\limsup_{u \searrow \sigma} \mathbf{1}_{u \in A^c} = 0$. We cannot have $L_\sigma > F_\sigma$ with positive probability, either, because then $\lim_{u \searrow \sigma} \mathbf{1}_{u \in A^c}$ would exist. Indeed, we would have

$$\tilde{\tau}(q) \geq \inf\{t \geq \sigma \mid F_t - F_\sigma \geq (L_\sigma - F_\sigma)/3 \text{ or } L_t - L_\sigma \leq -(L_\sigma - F_\sigma)/3\} =: \sigma'$$

for all $[q, \tilde{\tau}(q)) \cap (\sigma, \sigma') \neq \emptyset$ a.s. by $L_\sigma > \sup_{[\sigma, \sigma')}(L \wedge F)$, which means that there cannot be shorter and shorter intervals in A as we approach σ from the right. In summary, we only possibly affect right-continuity of $\alpha_i^\vartheta < 1$ by moving to $\tilde{\alpha}_i^\vartheta$ where $L = F$ and $\alpha_i^\vartheta > 0$, but

there $\alpha_i^\vartheta = 1$ by definition.

Finally, if we consider a (rationally valued, as above) sequence $\vartheta_n \searrow \tilde{\tau}(\vartheta)$, then (B.18) implies $\limsup_{u \searrow \tilde{\tau}(\vartheta)} \mathbf{1}_{u \in A^c} = 1$ a.s., and (B.17) implies $\tilde{\tau}(\vartheta) = \inf\{t \geq \tilde{\tau}(\vartheta) \mid \tilde{\alpha}_i^\vartheta(t) > 0\}$ a.s. by the definition of α_i^ϑ .

Now we can define $\tilde{G}_i^\vartheta := G_i^\vartheta$, $\tilde{G}_j^\vartheta := G_j^\vartheta$ as in Theorem 5.1 with $\tau^\vartheta := \tilde{\tau}(\vartheta)$ for any $\vartheta \in \mathcal{T}$. These strategies form a subgame-perfect equilibrium analogously to Theorem 7.3 with payoff $U_{(L \wedge F)^{\tilde{\tau}(\vartheta)}}(\vartheta) = U_{\tilde{L}}(\vartheta)$ at any $\vartheta \in \mathcal{T}$. \square

C Outcome probabilities

The following definition is a simplification of that in Riedel and Steg (2014), resulting from right-continuity of any $\alpha_i^\vartheta(\cdot)$ also where it takes the value 0.

Define the functions μ_L and μ_M from $[0, 1]^2 \setminus (0, 0)$ to $[0, 1]$ by

$$\mu_L(x, y) := \frac{x(1-y)}{x+y-xy} \quad \text{and} \quad \mu_M(x, y) := \frac{xy}{x+y-xy}.$$

$\mu_L(a_i, a_j)$ is the probability that player i stops first in an infinitely repeated stopping game where i plays constant stage stopping probabilities a_i and player j plays constant stage probabilities a_j . $\mu_M(a_i, a_j)$ is the probability of simultaneous stopping and $1 - \mu_L(a_i, a_j) - \mu_M(a_i, a_j) = \mu_L(a_j, a_i)$ that of player j stopping first.

Definition C.1. Given $\vartheta \in \mathcal{T}$ and a pair of extended mixed strategies $(G_1^\vartheta, \alpha_1^\vartheta)$ and $(G_2^\vartheta, \alpha_2^\vartheta)$, the *outcome probabilities* $\lambda_{L,1}^\vartheta$, $\lambda_{L,2}^\vartheta$ and λ_M^ϑ at $\hat{\tau}^\vartheta := \inf\{t \geq \vartheta \mid \alpha_1^\vartheta(t) + \alpha_2^\vartheta(t) > 0\}$ are defined as follows. Let $i, j \in \{1, 2\}$, $i \neq j$.

If $\hat{\tau}^\vartheta < \hat{\tau}_j^\vartheta := \inf\{t \geq \vartheta \mid \alpha_j^\vartheta(t) > 0\}$, then

$$\begin{aligned} \lambda_{L,i}^\vartheta &:= (1 - G_i^\vartheta(\hat{\tau}^\vartheta -))(1 - G_j^\vartheta(\hat{\tau}^\vartheta)), \\ \lambda_M^\vartheta &:= (1 - G_i^\vartheta(\hat{\tau}^\vartheta -))\alpha_i^\vartheta(\hat{\tau}^\vartheta)\Delta G_j^\vartheta(\hat{\tau}^\vartheta). \end{aligned}$$

If $\hat{\tau}^\vartheta < \hat{\tau}_i^\vartheta := \inf\{t \geq \vartheta \mid \alpha_i^\vartheta(t) > 0\}$, then

$$\begin{aligned} \lambda_{L,i}^\vartheta &:= (1 - G_j^\vartheta(\hat{\tau}^\vartheta -))(1 - \alpha_j(\hat{\tau}^\vartheta))\Delta G_i^\vartheta(\hat{\tau}^\vartheta), \\ \lambda_M^\vartheta &:= (1 - G_j^\vartheta(\hat{\tau}^\vartheta -))\alpha_j^\vartheta(\hat{\tau}^\vartheta)\Delta G_i^\vartheta(\hat{\tau}^\vartheta). \end{aligned}$$

If $\hat{\tau}^\vartheta = \hat{\tau}_1^\vartheta = \hat{\tau}_2^\vartheta$ and $\alpha_1^\vartheta(\hat{\tau}^\vartheta) + \alpha_2^\vartheta(\hat{\tau}^\vartheta) > 0$, then

$$\begin{aligned} \lambda_{L,i}^\vartheta &:= (1 - G_i^\vartheta(\hat{\tau}^\vartheta -))(1 - G_j^\vartheta(\hat{\tau}^\vartheta -))\mu_L(\alpha_i^\vartheta(\hat{\tau}^\vartheta), \alpha_j^\vartheta(\hat{\tau}^\vartheta)), \\ \lambda_M^\vartheta &:= (1 - G_i^\vartheta(\hat{\tau}^\vartheta -))(1 - G_j^\vartheta(\hat{\tau}^\vartheta -))\mu_M(\alpha_1^\vartheta(\hat{\tau}^\vartheta), \alpha_2^\vartheta(\hat{\tau}^\vartheta)). \end{aligned}$$

If $\hat{\tau}^\vartheta = \hat{\tau}_1^\vartheta = \hat{\tau}_2^\vartheta$ and $\alpha_1^\vartheta(\hat{\tau}^\vartheta) + \alpha_2^\vartheta(\hat{\tau}^\vartheta) = 0$, then

$$\begin{aligned} \lambda_{L,i}^\vartheta &:= (1 - G_i^\vartheta(\hat{\tau}^\vartheta -))(1 - G_j^\vartheta(\hat{\tau}^\vartheta -)) \frac{1}{2} \left\{ \liminf_{\substack{t \searrow \hat{\tau}^\vartheta \\ \alpha_i^\vartheta(t) + \alpha_j^\vartheta(t) > 0}} \mu_L(\alpha_i^\vartheta(t), \alpha_j^\vartheta(t)) \right. \\ &\quad \left. + \limsup_{\substack{t \searrow \hat{\tau}^\vartheta \\ \alpha_i^\vartheta(t) + \alpha_j^\vartheta(t) > 0}} \mu_L(\alpha_i^\vartheta(t), \alpha_j^\vartheta(t)) \right\}, \\ \lambda_M^\vartheta &:= 0. \end{aligned}$$

Remark C.2.

- (i) λ_M^ϑ is the probability of simultaneous stopping at $\hat{\tau}^\vartheta$, while $\lambda_{L,i}^\vartheta$ is the probability of player i becoming the leader, i.e., that of player j becoming follower. It holds that $\lambda_M^\vartheta + \lambda_{L,i}^\vartheta + \lambda_{L,j}^\vartheta = (1 - G_i^\vartheta(\hat{\tau}^\vartheta -))(1 - G_j^\vartheta(\hat{\tau}^\vartheta -))$. Dividing by $(1 - G_i^\vartheta(\hat{\tau}^\vartheta -))(1 - G_j^\vartheta(\hat{\tau}^\vartheta -))$ where feasible yields the corresponding conditional probabilities.
- (ii) If any $\alpha^\vartheta = 1$, then no limit argument is needed. Otherwise both α^ϑ are right-continuous and the corresponding limit of μ_M exists. μ_L , however, has no continuous extension at the origin, whence we use the symmetric combination of \liminf and \limsup , ensuring consistency whenever the limit does exist. If the limit in a potential equilibrium does not exist, both players will be indifferent about the roles; see Lemma A.5 in Riedel and Steg (2014).

References

- Bismut, J.-M. and B. Skalli (1977). Temps d'arrêt optimal, théorie générale des processus et processus de Markov. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 39, 301–313.
- Bulow, J. and P. Klemperer (1999). The generalized war of attrition. *Am. Econ. Rev.* 89(1), 175–189.
- Dellacherie, C. and P.-A. Meyer (1978). *Probabilities and Potential*, Volume 29 of *Mathematics Studies*. Amsterdam New York Oxford: North-Holland.
- Dutta, P. K., S. Lach, and A. Rustichini (1995). Better late than early: Vertical differentiation in the adoption of a new technology. *J. Econ. Manage. Strategy* 4(4), 563–589.
- Dutta, P. K. and A. Rustichini (1993). A theory of stopping time games with applications to product innovations and asset sales. *Econ. Theory* 3, 743–763.
- Fudenberg, D. and J. Tirole (1985). Preemption and rent equalization in the adoption of new technology. *Rev. Econ. Stud.* 52, 383–401.
- Fudenberg, D. and J. Tirole (1986). A theory of exit in duopoly. *Econometrica* 54(4), 943–960.
- Ghemawat, P. and B. Nalebuff (1985). Exit. *Rand J. Econ.* 16(2), 184–194.
- Hamadène, S. and M. Hassani (2014). The multi-player nonzero-sum Dynkin game in continuous time. *SIAM J. Control Optim.* 52(2), 821–835.
- Hamadène, S. and J. Zhang (2010). The continuous-time nonzero-sum Dynkin game problem and application in game options. *SIAM J. Control Optim.* 48(5), 3659–3669.
- Hendricks, K., A. Weiss, and C. Wilson (1988). The war of attrition in continuous time with complete information. *Int. Econ. Rev.* 29(4), 663–680.
- Hendricks, K. and C. Wilson (1992). Equilibrium in preemption games with complete information. In M. Majumdar (Ed.), *Equilibrium and Dynamics: Essays in Honour of David Gale*, pp. 123–147. Basingstoke, Hampshire: Macmillan.
- Hoppe, H. C. and U. Lehmann-Grube (2005). Innovation timing games: a general framework with applications. *J. Econ. Theory* 121, 30–50.
- Kallenberg, O. (2002). *Foundations of Modern Probability* (2nd ed.). New York, Berlin, Heidelberg: Springer.
- Lambrecht, B. and W. Perraudin (2003). Real options and preemption under incomplete information. *J. Econ. Dyn. Control* 27, 619–643.

- Laraki, R. and E. Solan (2013). Equilibrium in two-player non-zero-sum Dynkin games in continuous time. *Stochastics* 85(6), 997–1014.
- Laraki, R., E. Solan, and N. Vieille (2005). Continuous-time games of timing. *J. Econ. Theory* 120, 206–238.
- Mason, R. and H. Weeds (2010). Investment, uncertainty and pre-emption. *Int. J. Ind. Organ.* 28(3), 278–287.
- Mertens, J.-F. (1972). Théorie des processus stochastiques généraux: Applications aux surmartingales. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 22, 45–68.
- Murto, P. (2004). Exit in duopoly under uncertainty. *Rand J. Econ.* 35(1), 111–127.
- Pawlina, G. and P. M. Kort (2006). Real options in an asymmetric duopoly: Who benefits from your competitive disadvantage? *J. Econ. Manage. Strategy* 15(1), 1–35.
- Riedel, F. and J.-H. Steg (2014). Subgame-perfect equilibria in stochastic timing games. Working Paper 524, Center for Mathematical Economics, Bielefeld University.
- Smets, F. (1991). Exporting versus FDI: The effect of uncertainty, irreversibilities and strategic interactions. Working paper, Yale University, New Haven, CT.
- Steg, J.-H. and J. J. J. Thijssen (2015). Quick or persistent? Strategic investment demanding versatility. Working Paper 541, Center for Mathematical Economics, Bielefeld University.
- Thijssen, J. J. J. (2010). Preemption in a real option game with a first mover advantage and player-specific uncertainty. *J. Econ. Theory* 145, 2448–2462.
- Touzi, N. and N. Vieille (2002). Continuous-time Dynkin games with mixed strategies. *SIAM J. Control Optim.* 41, 1073–1088.
- Weeds, H. (2002). Strategic delay in a real options model of R&D competition. *Rev. Econ. Stud.* 69, 729–747.